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MATHEMATICAL MODELS OF THE REPARABLE ITEM INVENTORY SYSTEM

by

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ABSTRACT:

Reparable items, as opposed to consumable items, are usually rebuilt upon removal from service. A reparable item inventory system is composed of an inventory of ready-for-issue (RFI) items and an inventory of non-ready-for-issue (NRFI) items awaiting repair at the overhaul and repair facility. Since not all units issued in RFI condition will be recovered, procurement is necessary to supplement repair and maintain the population of units. This report describes such a system and presents a number of mathematical models, both deterministic and probabilistic, which prescribe the manner in which such a system should be operated.

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PREFACE

Research to "increase the Navy's understanding of the operations of inventory systems for reparable items" was initiated on 1 June, 1966, under NAVSUP RDT&E task area No. TF015-02-100. This report is the third stemming from research on this task; the first two are references [1] and [2]. The purpose of this document is to report and summarize efforts to structure mathematical models and optimal decision rules for such systems.

1. INTRODUCTION

1.1 Background

The Naval Supply Systems Command stocks a number of moderate to high cost items which are designated as "reparable". Upon failure a reparable item may be scrapped, but it is usually returned from the user to its designated overhaul and repair (O&R) point. After repair, the item is sent to the ready-for-issue (RFI) inventory to await demand. Items are designated reparable, as opposed to consumable, based on the feasibility of repair and the economics involved. Once an item has been so designated, it is presumably more economical to repair the item than it is to dispose of it and replace it with a new item. Decision rules for classifying items as reparable or consumable will not be considered here.

Classical inventory theory is a theory for consumable items. While the vast majority of all items held in inventory as stock is consumable items, the influence of reparable items may, nonetheless, be significant. At the Naval Aviation Supply Office reparables account for only seven per cent of the total items stocked, but account for 58 per cent of the dollars invested in inventory [3].

One can think of the "reparable system" as a loop containing three major organizations: the reporting stock point, the user (non-reporting stock point), and the O&R facility.* See Figure 1.** Losses occur in the system, however. Not every failed item will be returned to the O&R by the user, and the O&R cannot always economically repair an item returned to it by the user. These losses are replaced through the procurement of new items. The system contains two inventories. The reporting stock point inventory of RFI items is the primary inventory and is supplied from two sources, procurement and repair. The secondary inventory is the inventory of non-ready-for-issue (NRFI) items awaiting repair at the O&R. Given the reparable system with two inventories, the primary having two input sources with different lead times and costs, the problem concerns the specification of operating rules which will minimize the cost per unit time of operating the reparable item inventory system.

1.2 Literature

The majority of the published material in inventory theory is implicitly addressed to consumable items. Reparable item inventory systems have been studied primarily by the armed services. It is known that the problem was the subject of a student paper at the

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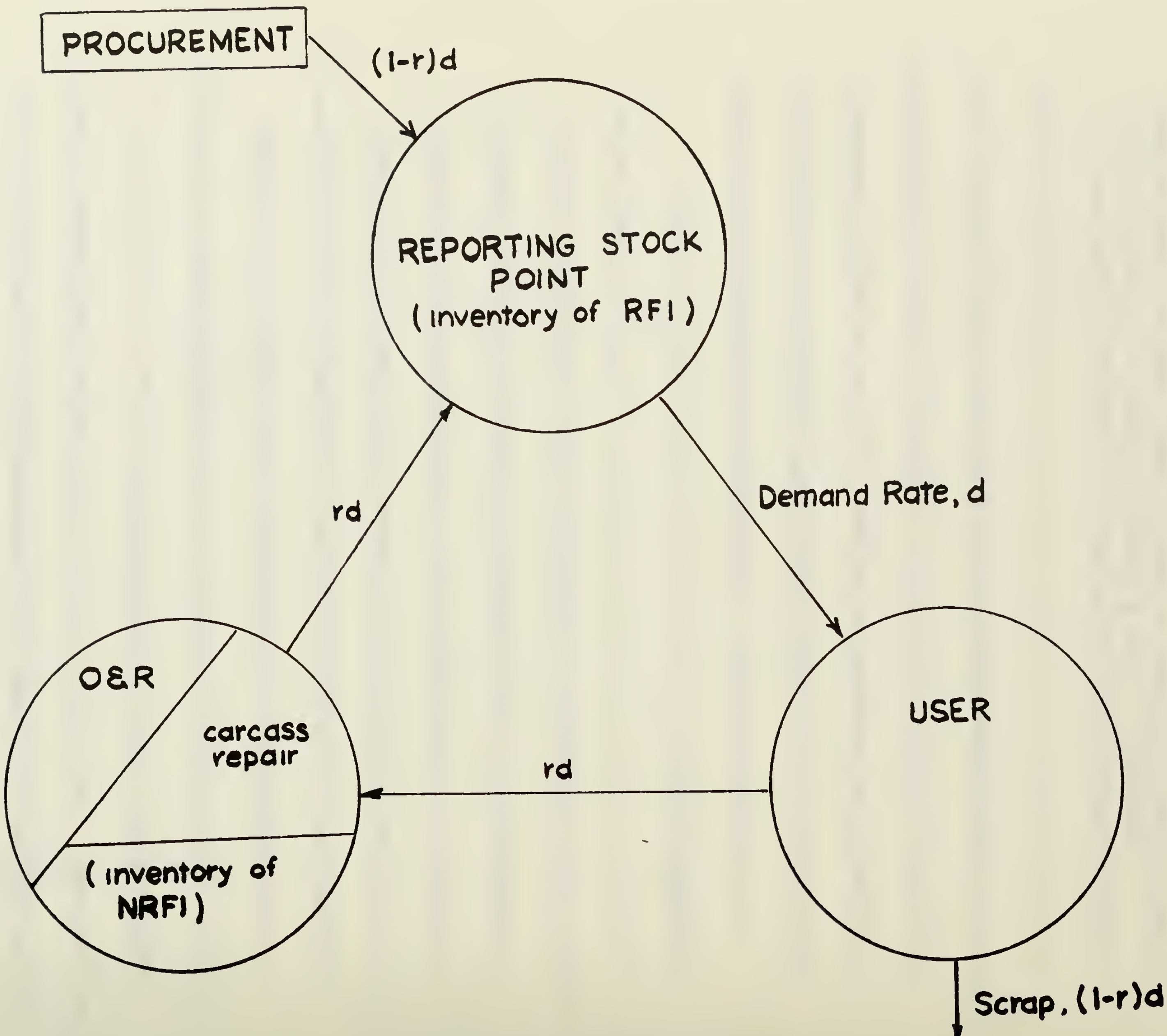
*Reparable items may be designated as repairable at the local, intermediate, or major level. This paper considers only the reparables designated for repair at the major level, the O&R level.

**Equations and figures are numbered within each section of the report, rather than consecutively through the report.



The Reparable Inventory System

FIGURE 1

Industrial College of the Armed Forces in 1939.* Analytic interest in the problem was initiated in the 1958-59 period by the services and has remained active since then.

Research on reparables has been performed for the Air Force by the RAND Corporation [4], [5], [6], [7], [8]. The culmination of this work [9] treats a two echelon (base-depot) supply system from the viewpoint of the base. The base is expected to perform the majority of the repair work. A fraction of the failed items are assumed reparable only at the depot level and these events trigger demands by the base for RFI items from the depot. Lateral resupply between bases is not considered. Only high-cost, low-demand items, for which unit order quantities are appropriate, are considered. Further, procurement is not considered, as complete recoverability is assumed.

Research on reparable item inventory systems has been performed for the Army by the Operations Research Center at the Massachusetts Institute of Technology (MIT) and by the Inventory Research Office of the Army Supply and Maintenance Command. The MIT research [10] treats a "two-stock policy". It is named the two-stock policy because it corresponds conceptually to a situation

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*The student involved was later to serve as Quartermaster General of the Army, Major General Kester L. Hastings. General Hastings' son, Captain David A. Hastings, USA, is one of the contributors to this report.

where there are two separate stock managers. The assumption is made that total demand for RFI items can be split into two classes of customers, those who return a NRFI item with their request for an RFI item and those who request an RFI item without returning a carcass in exchange. Thus, two inventories of RFI items are maintained and are managed independently.

The Inventory Research Office has noted that the procurement and repair decisions should not be made independently and has formulated what they call a "look ahead policy" in which procurement decisions are made with cognizance of the repair situation [11]. They have also addressed the questions of how many spares of a particular aircraft component should be purchased and how they should be distributed geographically [12]. Reference [13] is a qualitative statement of Army procedures for the management of reparable items.

Reparables research for the Navy has been performed by the Operations Analysis Department of the Fleet Material Support Office and the Naval Postgraduate School [1], [14], [15]. Additionally, the Office of Naval Research has sponsored a number of investigations which relate to the reparables problem [16], [17], [18]. These researches deal with inventory systems in which emergency orders, outside normal supply channels, may be placed to "bail out" the system from time to time. The emergency orders correspond to procurement orders and the normal channel of supply corresponds to repair.

1.3 Organization of the Report

Section 2 of the report presents a deterministic model and determines jointly the optimal procurement and repair quantities as well as their timing. The model is simple and approximate, yet it is useful as an introduction to the problem and in defining the form of policies. Until this is done, it is not at all clear as to how to "coordinate" the procurement and repair decisions.

Section 3 deals with a series of models of the reparable system in which demand for the RFI item is assumed to be a random variable with known density function. The first model treats repair as a continuous, vice batch, process and determines the optimal procurement order quantity - reorder point and repair rate. NRFI item returns to the O&R are also considered to be probabilistic. The three models which follow in this section were developed as master's degree theses under supervision of the author.

Subsection 3.2, by Lieutenant James E. Freiheit, USN, extends the deterministic model of Section 2 to the probabilistic case. Lieutenant Freiheit assumes continuous review policies for both procurement and repair and determines order quantities and reorder points for both.

The third model, Subsection 3.3, was formulated by Captain David A. Hastings, USA. Captain Hastings' thesis assumes that both the RFI demand rate and the NRFI item return rate are probabilistic. Procured RFI items and repaired RFI items are conceptually separated, though they are physically coordinated. Procurement operates with a periodic review policy while repair is operated with regularly scheduled inductions of all the NRFI item carcasses available at the time of an induction. The optimal review period and order up to level for procured items are determined along with the optimal time between repair batch inductions.

The final model of Section 3, due to Lieutenant Commander Paul A. Dollard, SC, USN, treats the reparable system as consisting of two subsystems: (1) the repaired item subsystem, and (2) the procured item subsystem. The demand rate and NRFI item return rate are assumed to be random variables with stationary normal probability density functions. Periodic review policies are assumed for both subsystems. Sufficient quantities are ordered at each review to bring both subsystems' inventory positions up to established levels. Net inventory distributions are developed for the inventory control point (ICP) repaired item inventory, the ICP procured item inventory, the ICP combined procured and repaired item inventory, and the repair facility inventory of NRFI items. A feature of the formulation is that expressions for system costs, in terms of the review periods and order up to levels, are developed as functions of the desired protection levels.

The final part of the report, Section 4, discusses the appropriateness of these models for current supply system operations, presents some conclusions and implications, and points out extensions and areas where further research seems profitable.

2. A DETERMINISTIC MODEL OF THE REPARABLE ITEM INVENTORY SYSTEM

2.1 Introduction

A reparable system is formulated in which all system parameters, including demand, are assumed known and constant. Back orders, which are avoidable in deterministic systems, are not permitted as the objective of military supply systems is the minimization of dis-service as measured by shortages. The formulation treats the RFI and NRFI item inventories as interdependent parts of a total system, and jointly determines the optimal procurement and repair quantities and their timing.

2.2 Basic Models

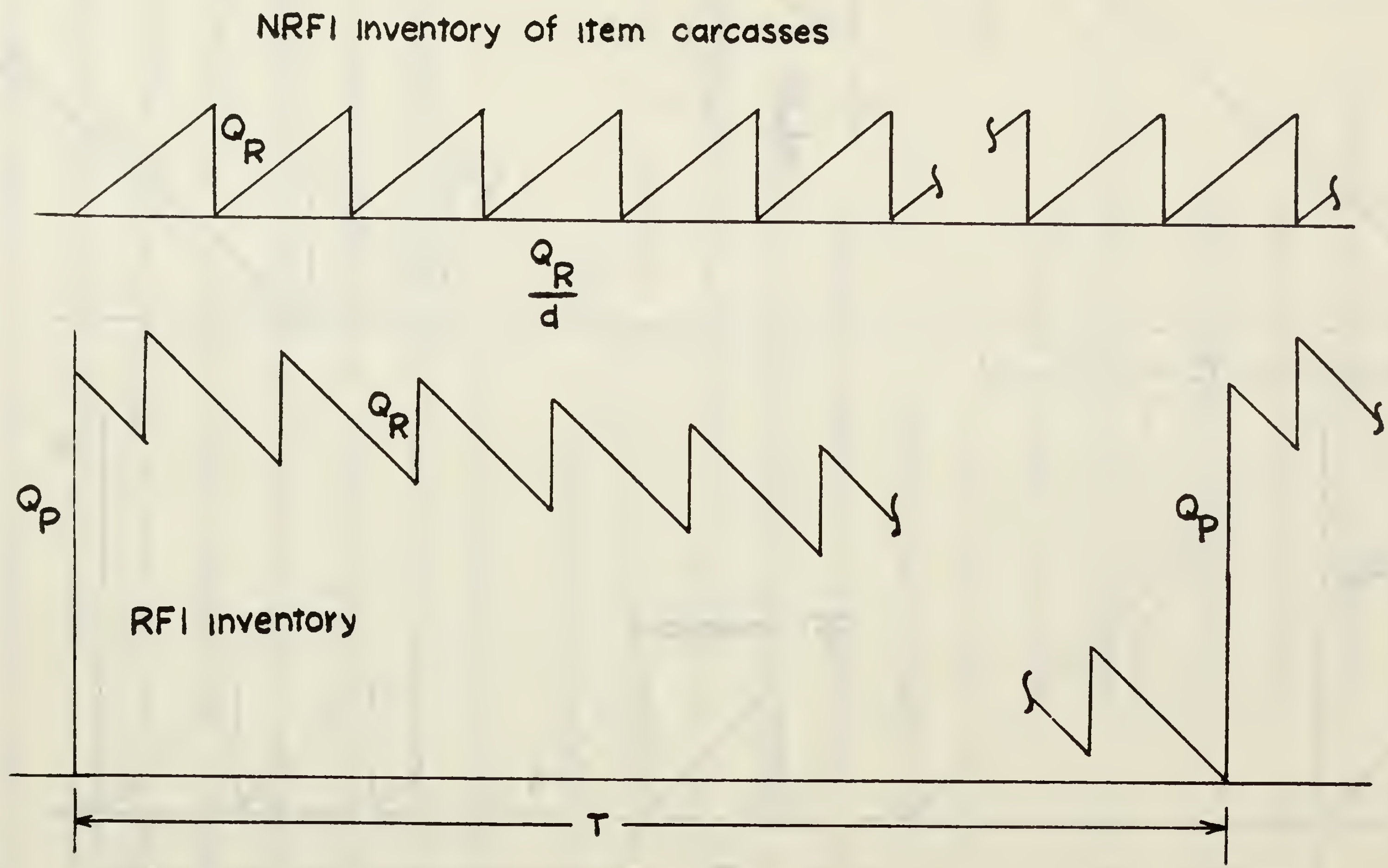
In attempting to formulate any sort of model of the reparable system, one is immediately impressed with the need for defining the reporting stock point of RFI stocking policy. An obvious objective is to supply as much of the total demand as possible with repaired RFI items, thus minimizing procurement. Beyond this, the reporting stock point must determine the scheme or policy of RFI inputs.

Two policies suggest themselves. The first policy calls for the O&R to induct a batch of carcasses when the NRFI inventory level reaches a certain point, the repair trigger. With a deterministic system, this rule will insure regularly spaced O&R inductions of fixed batch size and will be a simple procedure for the O&R to implement.

Batch sizes and their regularity can be maintained if there is continuous supplementing of repaired RFI with new procurement. The procurement order will be issued with enough lead time so that when the on-hand RFI stock drops to zero, a procurement quantity will arrive. The two inventories, RFI and NRFI, will have time histories as shown in Figure 2. Note that in this, the "continuous supplement" policy, the procurement trigger is in the RFI inventory and the repair trigger is in the NRFI inventory.

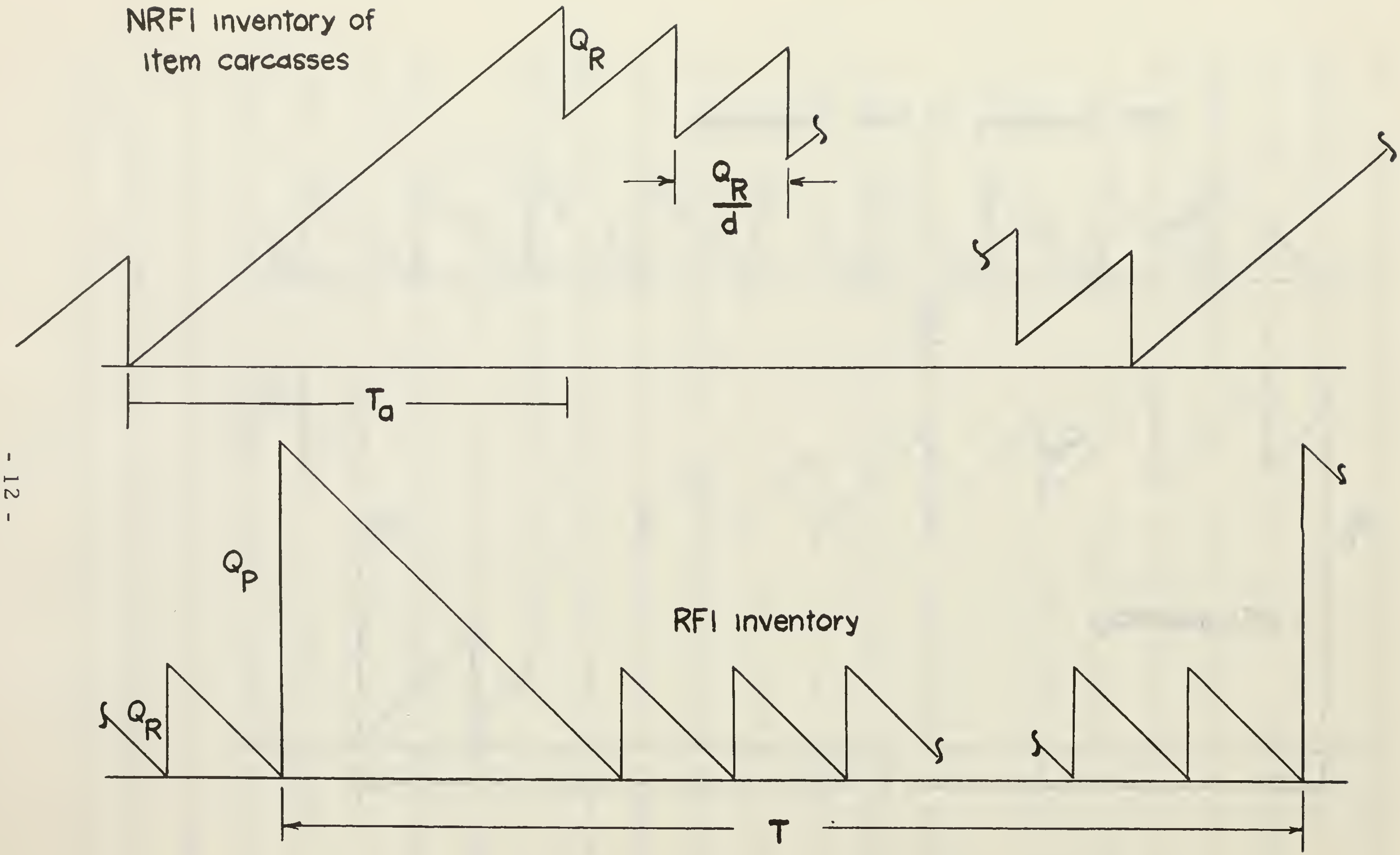
The second policy is suggested by noting that there is a trade-off between stock held in RFI condition and stock held in NRFI condition. Inventory is an idle resource. NRFI carcasses are a legitimate resource; they require only repair to be restored to full usefulness. But, the cost of this resource, NRFI items, is less than the cost of the RFI resource by at least the cost of repair labor and replacement parts. Thus, if inventory is to be held in the system it would be better held in NRFI condition than in RFI condition.

The second policy, the "substitution" policy, supplies 100 per cent of demand from repaired items until the supply of NRFI items decreases to a point where there are insufficient carcasses on hand to induct another batch. At this time, a procurement quantity is received, and O&R inductions are suspended. While the procurement quantity lasts, carcasses are accumulated at the O&R. Inductions are resumed a repair lead time before the procurement quantity is exhausted. The repair batch size is constant. The time histories of



The "Continuous Supplement" Policy

FIGURE 2



The "Substitution" Policy

FIGURE 3

the RFI and NRFI inventories for this policy are shown in Figure 3. Note that the repair trigger is in the RFI inventory and the procurement trigger is dependent upon the NRFI inventory, the reverse of the situation in the continuous supplement policy. The substitution policy minimizes the RFI inventory while the continuous supplement policy minimizes the NRFI inventory. It is the purpose of this report to study the substitution policy model. The continuous supplement model has been presented elsewhere [1].

2.3 The Substitution Policy Model

The model is formulated in terms of a single source of scrap. We assume that all carcasses inventoried by the O&R are reparable. This will simplify the model slightly and does not deter from its generality. We define the basic notation as follows:

- Q_P - procurement quantity;
- Q_R - repair batch size;
- d - demand rate, units per unit time;
- $(1 - r)$ - scrap rate, r is the recovery rate measured as a percent of the demand rate d;
- τ_P, τ_R - procurement and repair lead times;
- A_P - fixed procurement cost, per order;
- A_R - fixed repair batch induction cost, per batch;
- h_1 - RFI holding cost, per unit per unit time;
- h_2 - NRFI holding cost, per unit per unit time;
- T - system cycle time, time between successive procurement quantity arrivals to RFI inventory.

We will develop expressions for the total cost per cycle and the cycle time in terms of the two order quantities and known constants. From these expressions, we will obtain the total cost per unit time expression which will then be differentiated with respect to Q_P and Q_R to imply the two optimal order quantities.

Referring to Figure 3, define T_a as the time period during which inductions are suspended and the O&R is simply accumulating NRFI carcasses. By a variety of arguments, it may be determined that

$$T_a = (Q_P + Q_R) / d \quad . \tag{1}$$

Next, let n be the number of inductions per cycle. In general, it will not be possible to insure that the last induction before T_a begins will reduce the NRFI stock to zero. The residual NRFI stock when T_a begins will be some fraction of the net loss in NRFI items per repair cycle, where the repair cycle is the time between regular consecutive inductions. The net loss of NRFI inventory over the period between successive inductions is given by the rate of accumulation times the accumulation time, minus the induction quantity; i.e.,

$$rd\,(Q_R / d) - Q_R = - Q_R (1 - r) \ . \tag{2}$$

Thus, the residue when T_a begins will be some fraction of $Q_R(1 - r)$, call it $\beta Q_R (1 - r)$ where $0 \le \beta < 1$. While the β factor is unavoidable due to the requirement that there be an integral number of repair batches in each cycle, we shall set $\beta = 0$ in subsequent developments. The inaccuracies thus introduced are negligible as scrap rates, $(1 - r)$, are on the order of five to ten per cent, and the term $\beta Q_R (1 - r)$ is correspondingly small.

Now, the first batch after inductions are resumed takes the amount Q_R from the NRFI inventory. Subsequent inductions cause a net reduction of only $Q_R (1 - r)$ items. Thus, the amount of NRFI items available for the $(n - 1)$ inductions (all but the first) is $rd T_a - Q_R$, or, after simplification, $Q_R (r - 1) + r Q_P$. Dividing the last expression by $Q_R (1 - r)$, we obtain

$$(n - 1) = -1 + \left(\frac{r}{1 - r}\right) \frac{Q_P}{Q_R} \quad ,$$

or

$$n = \left(\frac{r}{1 - r}\right) \frac{Q_P}{Q_R} \quad . \tag{3}$$

The system cycle time, T, is then $(n Q_R + Q_P) / d$, or

$$T = Q_P / (1 - r) d \quad . \tag{4}$$

The total cost per cycle will be the fixed procurement order cost times the number of procurements per cycle (one), plus the fixed induction cost times the number of inductions per cycle (n), plus the RFI holding cost times the area under the RFI curve, plus the NRFI holding cost times the area under the NRFI curve. The area under the RFI curve is obtained as the sum of n triangles of height Q_R and base Q_R / d, plus the triangle of height Q_P and base Q_P / d.

See Figure 3. The area under the RFI curve, over one complete cycle, A_1, is then

$$A_1 = \frac{1}{2d} \left(\frac{r}{1 - r} \right) \left[Q_P Q_R + \left(\frac{1 - r}{r} \right) Q_P^2 \right] . \tag{5}$$

Referring again to Figure 3, the area under the NRFI curve is divided into a triangle and (n - 1) trapezoids.

The area of the triangle is $\frac{1}{2} T_a (rd\ T_a)$, which when using equation (1) reduces to $(r / 2d) (Q_P + Q_R)^2$. The total area of the (n - 1) trapezoids is determined in terms of a quantity L, the smaller height of the largest trapezoid. The larger heights are related to the smaller heights by the constant $r Q_R$. Successive lower heights are related by the constant $Q_R (r - 1)$. It may be verified, using the above relations and the relation that the sum of the first $k - 1$ positive integers equals $\frac{k(k - 1)}{2}$, that the total area of k such trapezoids is given by the expression

$$\frac{k Q_R}{2d} \left[2L + r Q_R - (k - 1) Q_R (1 - r) \right] .$$

Finally, for the model, $L = rd\ T_a - Q_R$, which reduces to $r Q_P - Q_R (1 - r)$ after utilizing equation (1). The total area under the NRFI curve, A_2, is then, after simplification,

$$A_2 = \left(\frac{1}{2d} \right) \left(\frac{r}{1 - r} \right) \left[Q_P Q_R + Q_P^2 \right] . \tag{6}$$

We may now write down the total cost per cycle expression:

$$TC / \text{cycle} = A_P + n A_R + h_1 A_1 + h_2 A_2 \quad . \tag{7}$$

Using the expressions for n, A_1, and A_2, and dividing equation (7) by the system cycle time, $T = (Q_P + n Q_R) / d = Q_P / d(1 - r)$, yields our objective -- the total cost per unit time expression:

$$TC/t = \frac{A_P d(1-r)}{Q_P} + \frac{A_R rd}{Q_R} + \frac{h_1 r}{2}\left[Q_R + \left(\frac{1-r}{r}\right) Q_P\right] + \frac{h_2 r}{2}\left[Q_P + Q_R\right] . \tag{8}$$

The optimal order quantities are obtained by setting the partial derivatives, with respect to Q_P and Q_R, of equation (8) equal to zero and solving the resulting equations. The optimal order quantities are

$$Q_P^* = \sqrt{\frac{2 A_P d(1-r)}{h_1(1-r) + h_2 r}} , \tag{9}$$

and

$$Q_R^* = \sqrt{\frac{2 A_R d}{h_1 + h_2}} . \tag{10}$$

Equations (9) and (10) represent simply the appropriate modifications of the EOQ formula of elementary inventory theory. Also note that if the scrap rate is zero, $r = 1$, then there is no need to procure new items in the system formulated and the solution of equation (9) sets $Q_P^* = 0$.

2.4 An Example

The following example is used to indicate the nature of the solutions given by the model. The parameter values were arbitrarily chosen, but are thought to be at least representative. The parameter values used were as follows: $A_P = \$750$, $A_R = \$100$, $r = 0.9$, $d = 1{,}000$ units per year, $h_1 = \$200$, and $h_2 = \$20$ -- based on a unit cost of $1,000, $\tau_P = 1.0$ year, and $\tau_R = 0.25$ years. With these values, the following results obtain (with some rounding off): $Q_P = 63$, $Q_R = 30$, $n = 19$, and $T = 0.633$ years. There will be $r(Q_R + Q_P) \doteq 84$ carcasses available for repair at the end of T_a. The initial induction will reduce this amount by 30 units, leaving 54 units. Further inductions result in a net loss of NRFI inventory of $Q_R(1 - r) = 3$ units. Thus, the 54 units will be sufficient for 18 batches. These 18, plus the initial batch, make a total of 19 repair batches per cycle, as indicated above. The annual cost of operating the inventory system is $8,345.*

Finally, we comment on the reorder points δ_P and δ_R. In the deterministic system studied, the reorder points were not explicitly considered. However, we must be careful to choose δ_P

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*We note that the continuous supplement policy of reference [1], applied to this example, yields an annual cost of $11,070.

so as to have the procurement arrive at the moment that the last repaired batch before T_a is depleted. If

$$(m - 1)T \leq \tau_P < mT, \quad m \geq 1 ,$$

then set $\delta_P = mT - \tau_P$ time units after a procurement arrival. In the example $T < \tau_P < 2T$, so set $\delta_P = 2T - \tau_P = 0.266$ years after each procurement arrival.

While the procurement reorder point is most conveniently determined in terms of time since a procurement arrival, because of the ambiguities a reorder point quantity would present, the repair reorder point is determined in terms of the RFI units on hand. For this example $\delta_R = 10$ RFI units on hand, except when the time until a procurement arrival is first less than τ_R, inductions are suspended for a time T_a, and then resumed on a regular basis.

2.5 Conclusion

The model has been developed in the context of reparable items, whether in a military or civilian supply system. Application in other contexts is also possible. With appropriate modifications, the model could be used in a manufacturing situation where the maximum production rate is less than the demand rate, necessitating periodic purchases from outside sources to supplement production.

3. PROBABILISTIC MODELS

3.1 A Reparable Item Inventory System with Continuous Repair

3.1.1 Introduction

In this subsection, we postulate and structure a probabilistic model of the reparable item inventory system in which NRFI item repair is accomplished continuously, rather than in batches. There is a single RFI item inventory consisting of both procured RFI items and repaired RFI items. This inventory is replenished continuously with repaired RFI items and is augmented at discrete points in time by order quantities of procured RFI items. The input of new items is required because a percentage of the RFI items demanded will not be repaired and returned to the system in RFI condition; repaired RFI items alone cannot completely satisfy demand. A continuous review of the RFI inventory is assumed. A procurement quantity, Q_P, is ordered when the RFI inventory reaches the reorder level, δ_P. The procurement order quantity and reorder level are decision variables.

Operation of the repair facility is viewed in terms of a queueing system in which the customers are NRFI items requiring repair and the service is, of course, repair. The rate at which repair can be accomplished is some function of the resources (manpower, facilities, etc.) allocated to the repair of a particular item. Thus, there is a cost associated with the level of repair capacity μ. Increasing the repair capacity will increase O&R operating costs, but will reduce repair turn-around time and the queue (inventory) of NRFI items awaiting repair. Repair capacity, therefore, is also a decision variable in this formulation.

The formulation assumes the usual order and holding costs, along with a fixed cost per shortage and a cost per unit of repair capacity. The measure of effectiveness from which the optimal values of the decision variables will be determined is the minimization of total cost per unit time.

3.1.2 Notation and Further Assumptions

We define the following notation:

Q_P - procurement order quantity, a decision variable;

δ_P - procurement reorder point in terms of the inventory position (on hand plus on order minus back orders), a decision variable;

μ - repair capacity in units per unit time, assumed strictly greater than the mean NRFI item return rate, a decision variable;

x - RFI item demand, a random variable with stationary density $g(x;t)$ for the demand in time t, and mean $\bar{x}t$;

y - NRFI item return (to the O&R), a random variable with density $h(y;t)$ for the return in time t, and mean $\bar{y}t$;

z - net RFI item demand, a random variable with density $f(z;t)$ for the net demand in time t, and mean $\bar{z}t$;

τ_P - procurement lead time, assumed constant;

A_P - fixed procurement order cost;

h_1 - RFI item holding cost, dollars per unit per unit time;

h_2 - NRFI item holding cost, dollars per unit per unit time;

Π - shortage cost, dollars per unit short;

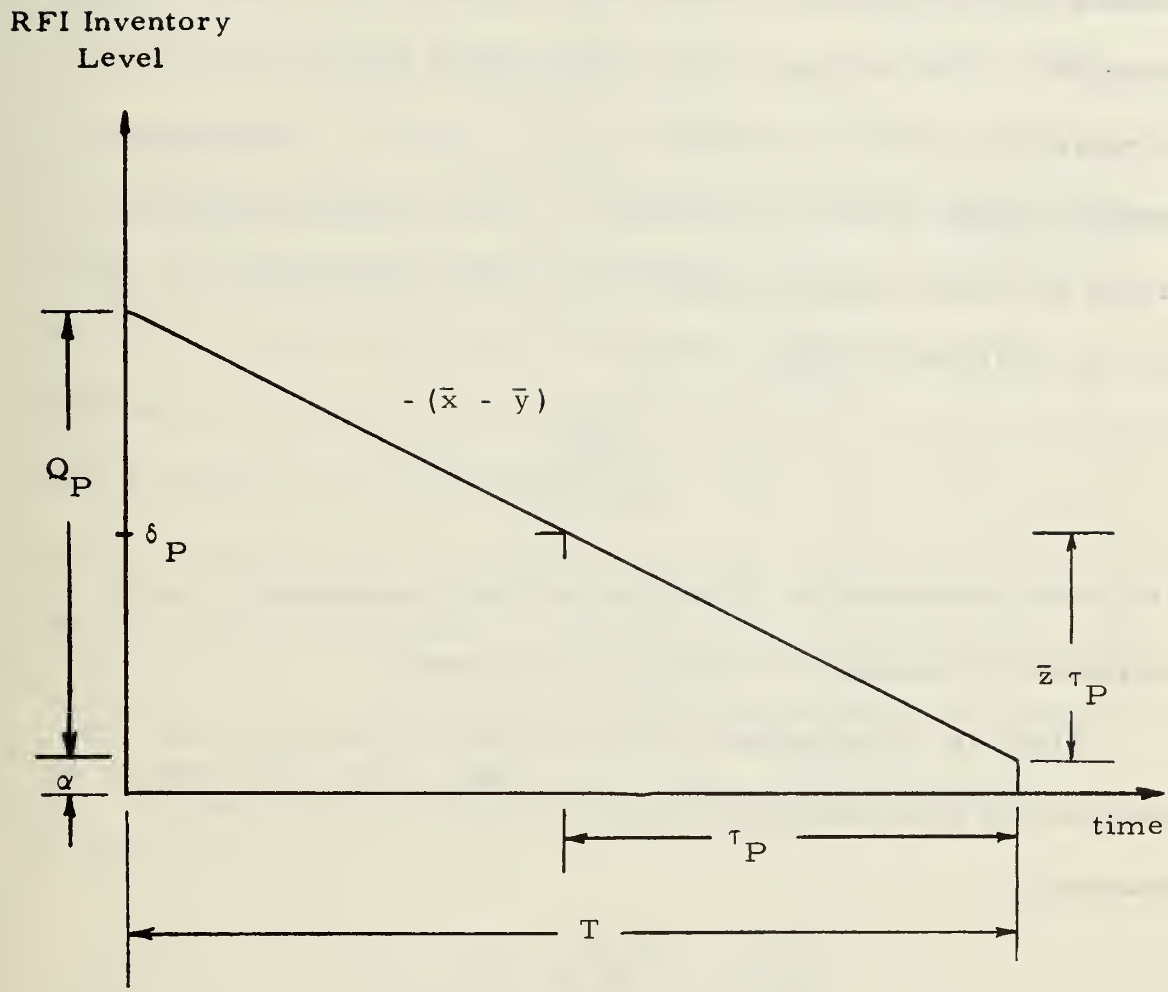
C_R - cost per unit of repair capacity, assumed linear at least in the range $\mu > \bar{y}$, so that the cost per unit time of maintaining a repair facility with capacity to repair μ units per unit time is simply $C_R \mu$;

T - expected cycle time, defined as the time between successive arrivals of procurement order quantities;

K - expected total variable cost per unit time.

3.1.3 The Model

The model is approximate in that certain of the cost components are determined by treating the expected values of random variables as time averages. The time history of the RFI item inventory through an "average" cycle is shown in Figure 1. RFI items are demanded at an average rate of $\bar{x}$ units per unit time. Since we will insist that the repair capacity exceed the mean carcass return rate to the O&R, the mean O&R output of repaired RFI items will be equal to the mean carcass return rate $\bar{y}$. The repaired RFI items from the O&R replenish the RFI item inventory continuously and at an average rate of $\bar{y}$ RFI items per unit time. Because the recovery rate is assumed strictly less than one, the O&R output of repaired RFI items will be less than the RFI item demand rate; $\bar{y} < \bar{x}$. The RFI item inventory thus experiences an average "net" demand of $(\bar{x} - \bar{y})$ items per unit time.



The RFI Item Inventory Over an "Average" Cycle

FIGURE 1

The total cost per unit time will be the sum of procurement order costs (POC), RFI item holding costs (RFI HC), NRFI item holding costs (NRFI HC), repair facility costs (RFC), and shortage costs (SC). Each of these costs is determined as a function of one or more of the decision variables Q_P, δ_P, and μ. We begin by determining the expected cycle time, T. The expected cycle time is just the time required to demand Q_P units at an average rate of $(\bar{x} - \bar{y})$ units per unit time. Thus,

$$T = \frac{Q_P}{(\bar{x} - \bar{y})} \quad . \tag{1}$$

The above expression for T is most certainly approximate, but is acceptable if shortages are incurred infrequently.

There is, by definition, one procurement order per cycle. The procurement order cost per unit time is $\frac{A_P}{T}$ or, upon using equation (1),

$$POC = \frac{A_P (\bar{x} - \bar{y})}{Q_P} \quad . \tag{2}$$

The holding costs are proportional to the units per unit time held in inventory. If we again appeal to the case where shortages are rare, the RFI holding cost per cycle is

$$h_1 \left[\frac{Q_P T}{2} + \alpha T \right] \quad .$$

But the safety stock, α, is $\delta_P - \bar{z}\,\tau_P$, where $\bar{z}\,\tau_P$ is the mean leadtime net demand. Substituting for α and dividing by the cycle time T, we obtain

$$\text{RFI HC} = h_1 \left[\frac{Q_P}{2} + \delta_P - \bar{z}\,\tau_P \right] . \tag{3}$$

Shortages will occur if leadtime demand exceeds the reorder level. Any reasonably significant shortage cost, Π, will dictate a reorder level which is larger than the mean leadtime demand. A deterministic analysis using expected values under these circumstances would indicate that no shortages occur. Yet we know that with probabilistic demand shortages are bound to occur from time to time. We see that if the net demand, z, exceeds δ_P, the shortages per cycle will be $(z - \delta_P)$. The expected number of shortages per cycle is the weighted sum of the quantities $(z - \delta_P)$ over all values of $z > \delta_P$; i.e.,

$$\int_{\delta_P}^{\infty} (z - \delta_P)\, f(z\,;\,\tau_P)\, dz \quad .$$

Hence, the shortage cost per unit time expression is

$$SC = \frac{\Pi\,(\bar{x} - \bar{y})}{Q_P} \int_{\delta_P}^{\infty} (z - \delta_P)\, f(z\,;\,\tau_P)\, dz \quad . \tag{4}$$

On the NRFI side of the system, the cost per unit time of establishing and operating a repair facility with capacity to repair

μ units per unit time was postulated to be $C_R \mu$. Thus,

$$RFC = C_R \mu \quad . \tag{5}$$

With continuous repair, we may think of the arrival of a NRFI item at the O&R as the arrival of a customer to a queueing system in which the service is repair. The inventory of NRFI items awaiting repair is then the number of customers in the queue awaiting service. The NRFI item holding cost per unit time may be written as $h_2 \ E(n)$, where $E(n)$ is the expected number of NRFI items in the repair facility. The holding cost is applied regardless of whether the item is waiting to be repaired or is actually being repaired. Until the unit actually enters the RFI item inventory, it is unfit for issue and therefore is NRFI. Since some charge is applicable to the holding of an asset at all times, the NRFI holding cost is applied to the item from the time it reaches the O&R until it reaches RFI condition.

Some assumptions are necessary about the queueing system which represents the repair facility. The most desirable, least restrictive, assumption about the carcass return and repair processes is that they constitute a queueing system with general, independent arrivals, general service, and a single server. The stationary distribution of the number in the system is theoretically available through a Wiener-Hopf type integral equation for the waiting time distribution. However, such representations cannot usually be reduced to closed form analytic expressions. Consequently, the slightly more restrictive, but workable, assumption is made that carcass return and repair processes constitute a queueing system with Poisson arrivals, general service, and a single server.

Another question which arises is whether the repair process should be modeled as a multiserver repair operation or as a facility (single server) with a certain repair capacity. Depending upon the nature of the reparable item, a case can probably be made for either viewpoint. Discussion of this point is motivated by differences that arise in the queueing theory representations. All other things being equal, a customer (NRFI item in need of repair) will wait longer but be serviced quicker in a single server queueing system than in a multiserver queueing system of the same total service capacity (maximum repair rate in units per unit time serviced). However, as indicated above, we are interested in the time spent in the repair system, and either queue description is satisfactory for our purpose.

With the Poisson arrival, general service, single server assumption, the Pollaceck-Rhintchine results are applicable; namely,

$$E[n] = \rho + \frac{\rho^2 + \lambda^2 \sigma^2}{2(1-\rho)} \quad ,$$

where ρ is the ratio of the mean arrival rate to the mean service rate, λ is the mean arrival rate, and σ^2 is the variance of the service time distribution. For the repair system being formulated, the mean NRFI item input to the O&R is $\bar{y}$ units per unit time. The mean service rate was defined as μ units per unit time. Thus, $\rho = \frac{\bar{y}}{\mu}$. Substituting these relations and simplifying, we obtain

$$E[n] = \frac{\bar{y}(2\mu - \bar{y} + \mu^2 \bar{y} \sigma^2)}{2\mu(\mu - \bar{y})} .$$

We may now write down the NRFI holding cost expression:

$$\text{NRFI HC} = \frac{h_2 \bar{y}(2\mu - \bar{y} + \mu^2 \bar{y} \sigma^2)}{2\mu(\mu - \bar{y})} . \qquad (6)$$

The total cost per unit time, K, is the sum of equations (2) through (6):

$$K = \frac{A_P(\bar{x} - \bar{y})}{Q_P} + h_1 \left[\frac{Q_P}{2} + \delta_P - z_P \tau_P \right] + C_R \mu$$

$$+ \frac{h_2 \bar{y}}{\mu} \left[\frac{2\mu - \bar{y} + \bar{y}\mu^2 \sigma^2}{2(\mu - \bar{y})} \right]$$

$$+ \frac{\Pi(\bar{x} - \bar{y})}{Q_P} \int_{\delta_P}^{\infty} (z - \delta_P) f(z; \tau_P)\, dz . \qquad (7)$$

The values of Q_P, δ_P, and μ which yield minimum K are the solutions of the equations

$$\frac{\partial K}{\partial Q_P} = \frac{\partial K}{\partial \delta_P} = \frac{\partial K}{\mu} = 0 .$$

The partial with respect to Q_P leads to

$$Q_P{}^* = \left[\frac{2(\bar{x} - \bar{y})}{h_1} \left\{ A_P + \Pi \int_{\delta_P}^{\infty} (z - \delta_P)\, f(z;\tau_P)\, dz \right\} \right]^{\frac{1}{2}} . \qquad (8)$$

The optimal reorder point is determined from

$$1 - F(\delta_P ; \tau_P) = \frac{Q_P h_1}{\Pi(\bar{x} - \bar{y})} , \qquad (9)$$

where $F(\delta_P ; \tau_P)$ is the cumulative distribution function of net leadtime demand evaluated up to δ_P. Equation (8) yields Q_P in terms of δ_P and equation (9) has δ_P in terms of Q_P. Consequently, equations (8) and (9) must be solved iteratively for $Q_P{}^*$ and $\delta_P{}^*$. The partial of K with respect to μ yields μ^* directly as the value of $\mu > \bar{y}$ which is a root of the fourth order equation

$$\mu^4 (2 C_R) - \mu^3 (4 C_R \bar{y}) + \mu^2 (2 C_R \bar{y}^2 - 2 h_2 \bar{y} - h_2 \bar{y}^3 \sigma^2)$$
$$+ \mu (2 h_2 \bar{y}^2) - h_2 \bar{y}^3 = 0 . \qquad (10)$$

A final note on the model is that the distribution of net demand will depend upon the parameters of the queueing system which represents the repair facility. If the utilization factor of the repair facility is strictly less than one (the only case which makes sense here), then the mean output from the repair facility will be the mean input, $\bar{y}$. However, strictly speaking, the repair output will be a random variable, say w, with mean $\bar{y}$ and distribution and second moments dependent upon the queueing system. The only well-known result about the output of a queue is that if arrivals have a Poisson distribution and service has an exponential distribution, then the output w has a Poisson distribution.

In any case, to be correct, the net RFI item demand z is equal to the RFI item demand minus the input of repaired RFI items from the repair facility;

$$z = x - w \quad .$$

For a utilization factor less than one, $\bar{z} = \bar{x} - \bar{y}$, as discussed above. However, the distribution of z and its variance will depend on the distribution of w and, of course, x. Any explicit results would thus require specification of such distributions.

3.1.4 An Example

The following example indicates the nature of the solutions to equations (8), (9), and (10). The parameter values are the same as those used in the example of Section 2, with some additions, of course. We assume $A_P = \$750$, $h_1 = \$200$ per RFI item per year, $h_2 = \$20$ per NRFI item per year, $\tau_P = 1.0$ years, $C_R = \$900$ per unit of repair capacity, and $\Pi = \$1,000$ per unit short. We further assume that RFI item demand has a normal distribution with mean of $1,000t$ and variance of $1,000t$, where time is measured in years. The NRFI return is assumed to have a Poisson distribution with mean and variance of $900t$, where t is measured in years. Further, we assume a constant repair time for simplicity.

It remains to specify the distribution of the net RFI item demand, z. The repair capacity μ, as determined from equation (10), is very close to the mean NRFI item return rate (shown below). This implies a utilization factor which is very nearly one, meaning that the repair facility is almost never idle. The effect is that the repair facility output is very nearly constant. Thus, $\bar{w} = \bar{y}$, and $Var(w) \doteq 0$. Then, since $z = x - w$ and since $\bar{y}$ is $900t$, it follows that z is normally distributed with mean $100t$ and variance $1,000t$.

The solution procedure is to first determine the roots of equation (10), setting μ^* equal to the positive root which exceeds $\bar{y}$. In this case, the roots of equation (10) are 0 ± 3.1623 and 900 ± 3.1623. As $\bar{y} = 900$ in this example, we set $\mu^* = 903.1623$. A computer routine is used to find the roots of equation (10). This value of μ produces a utilization factor in excess of 99 per cent, and hence the variance of net demand is as postulated.

We now must solve equations (8) and (9) for the optimal values of Q_P and δ_P. The iterative solution of these equations is accomplished with the same program used to solve equation (10). The iterations begin by setting

$$Q_P^{(1)} = \left[\frac{2(\bar{x} - \bar{y}) A_P}{h_1} \right]^{\frac{1}{2}}$$

and substituting this value into equation (9). Equation (9) is solved for $\delta_P^{(1)}$, which is then substituted into equation (8). Equation (8) is then solved for the new order quantity, $Q_P^{(2)}$, and the iterations continue until convergence is achieved.

Table 1 indicates the convergence of Q_P and δ_P through 10 iterations (the program went through a large number of iterations). The expected cycle time T, for this example, is 0.556 years or about 29 weeks. Annual total costs are predicted to be $837,470. This figure seems overwhelming and is difficult to compare with the costs produced by the models which follow (which also use the same basic parameters). If the cost of establishing and operating the repair facility, $C_R \mu$, is subtracted from the total cost as given by equation (7), we obtain an annual cost figure of $24,767, which is comparable to the results of the models which follow.

The above example was hypothetical, although some of the parameter values were thought to be representative. In particular, it was most difficult to select a value for C_R which seemed reasonable. Consequently, several values of C_R were considered. The results, in terms of the total cost and optimal values of the decision variables, are insensitive to C_R. A C_R of $720 yields $\mu^* = 903.536$, $Q_P^* = 55.677$, $\delta_P^* = 153.150$, and total annual

$C_R = \$900, \quad \mu = 903.162$

Q_P	δ_P	K
27.3861	169.7530	$838827.06
41.9823	160.1044	837695.29
49.2353	156.2436	837511.99
52.6852	154.5530	837476.66
54.2954	153.7913	837469.47
55.0406	153.4443	837467.98
55.3842	153.2854	837467.66
55.5423	153.2126	837467.60
55.6150	153.1791	837467.58
55.6485	153.1637	837467.58

$K - C_R \mu = \$24,760, \quad T = .556$ years

An Example of the Iterative Computations of Q_P and δ_P

TABLE 1

costs of \$674, 866. If $C_R \mu$ is subtracted from this amount, the annual cost is \$24, 325. If C_R = \$360, we obtain $\mu^* = 905.000$, $Q_P^* = 55.677$, $\delta_P^* = 153.151$, and total annual costs of \$349, 375, which less $C_R \mu$, is \$23, 575.

3.1.5 Conclusion

The model is thought appropriate for classes of reparable items which require extensive and special-purpose repair and test facilities. If the item also has a moderately large demand, this formulation seems to be a natural way of thinking of the operation of a reparable item inventory system. We also note that the examples have indicated that high repair facility utilization rates, which are of course good for the efficient operation of the repair facility, are also good for the reparable system.

Further study is required on the repair facility cost parameter, C_R. C_R was defined, very loosely, as the cost per unit of repair capacity. Further, $C_R \mu$ has been referred to as the cost per unit time of establishing and maintaining a repair facility with capacity to repair μ units per unit time.

The O&R activity at NAS Alameda recently established an clean room facility for the repair of hydraulic aircraft components. These components represented "new business" for the O&R since, for lack of proper facilities, the components previously had to be repaired by their manufacturer. It is in this context that the cost $C_R \mu$

was postulated as appropriate. Further study is required, however, to determine the relevant cost components of C_R (i. e., labor, overhead, depreciation on the facility, etc.). The problem of estimating C_R seems also to belong to another problem, production and facilities planning. Presumably something like C_R or $C_R \mu$ was estimated before the decision was made that authorized the clean room facility and the new hydraulic component repair capability at O&R Alameda.

3.2 A Continuous Review Model of the Reparable Item Inventory System

3.2.1 Introduction

The reparable inventory system differs from that of the classical textbook type in two major ways. First, the system contains two distinct inventories, one of which has as its members items that are in a ready-for-issue (RFI) state and the other containing items which are in a non-ready-for-issue (NRFI) state; i. e., the first contains items which are usable and the second contains items that must be repaired before they can be put to use. Second, the RFI inventory is made up of a mixture of new items and items that have been used, failed, repaired, and are ready to be used again.

The standard approach to the problem of how to run an inventory system has been to consider one in which there is a sole source of supply. In the reparable inventory system, this situation in general does not exist. Here the usable inventory may be thought of as being supplied by two separate processes which differ considerably from each other, the first being the manufacturer and the second being a repair facility. These processes may differ in their lead times and in the cost of producing a usable item. In general, the repair lead time is shorter than the procurement lead time, and the cost of repair is less than the cost of manufacturing. The arguments here tend to favor the repair facility as a primary source of inventory, but the savings in operating this facility must be large enough to warrant its existence. The criteria that an item must meet to mark it as reparable are threefold. First, the item must be physically capable of repair; second, the cost of accomplishing this repair must be considerably less than the purchase price of a new item; and third, the initial cost of the item must be large enough to warrant the cost associated with getting the carcasses back to the NRFI. Assuming that a repaired item is functionally identical to one which has never been used, it would seem that the repair facility is a more desirable supplier than the manufacturer and that the decision-maker should select only the facility to provide his inventory. This, in fact, would be the case if the system were one in which there were no losses and all carcasses were returned and met the criteria of being reparable. For example, if cost of repair is the criterion, then the cost of repairing some of the items might be considerably higher than the manufacturing cost. In this example, the rational thing for the decision-maker to do would be to discard those items that do not meet the criterion and replace them by procuring from the first supplier. Now assuming that the above is the case, then an examination of the system would indicate that a certain fraction (r) of the items demanded from the RFI inventory will eventually return to the RFI inventory.

Section 2 dealt with a deterministic model based on a "substitution" policy. This policy requires the repair facility to supply all the items demanded until the NRFI inventory drops to a level below that necessary for another repair batch induction. The next quantity that arrives at the RFI inventory is that which had previously been ordered through procurement. The demands are then satisfied with this procured quantity, while the NRFI inventory builds up and the cycle starts again. This paper will be devoted to this substitution policy, treating demand as a stochastic variable.

3.2.2 Notation and Assumptions

The "substitution" policy deals with a system that is self-sustaining to a certain point, at which time a procurement must arrive in order to compensate for the losses $((1 - r)\%)$ to the system. Since the repair facility is the primary source of usable inventory, the model is developed to determine the most feasible batch size and the number of batches per cycle that should be supplied by this process. The model is developed with the measure of effectiveness being the cost of operating the system per year, using the following notation:

Q_P - fixed procurement quantity, a decision variable;

Q_R - fixed repair batch size, a decision variable;

X - demand rate, units per unit time, a random variable with mean $\bar{X}$ and density $f(x, t)$ where t is a particular time period under consideration;

r - recovery rate, a given constant;

τ_P - procurement lead time, a given constant;

τ_R - repair lead time, a given constant;

A_P - fixed cost of placing a procurement order, per order, given;

A_R - fixed cost associated with an induction at the repair facility, per induction, given;

h_1 - RFI holding cost, dollars per unit per year, given;

h_2 - NRFI holding cost, dollars per unit per year, given;

n - number of inductions per cycle, a decision variable, $n = 0, 1, 2, \ldots$;

Π - cost of incurring back orders, dollars per unit, given;

T_R - expected time it takes for Q_R items to be demanded;

δ_P - procurement reorder level, based on inventory position, a decision variable;

δ_R - repair induction reorder level, based on inventory position, a decision variable;

T - system cycle time, a random variable, time between successive procurement quantity arrivals to RFI inventory;

TC_T - total cost of operating the system per cycle;

TC - total cost of operating the system per year.

The "real" world situation is a most difficult thing to express in terms of mathematical symbols. Here, as in most cases, assumptions have to be made in order to develop a model that, although only an approximation of the "real" world, gives an indication of how the "real" world behaves. The system considered in this paper is one which operates on a cyclic basis, where a cycle is the random amount of time (T) between the arrivals into the RFI inventory of procurement orders. The cycle will consist, therefore, of the arrival of one procurement and n repair inductions. As long as the distribution of the stochastic demand remains the same from one cycle to the next, an analysis of one cycle will describe how every other cycle behaves. The model presented considers a system whose inventory is made up of one type of item. The items that are repaired are considered equal in all respects to those procured. The model will be a continuous review-type based on an inventory position, (IP), where

$$IP = \text{inventory on hand} + \text{on order} - \text{back orders},$$

and the on-hand inventory is a non-negative quantity. The procurement and repair orders are placed when the IP drops to certain levels (the reorder levels). A fixed procurement order (Q_P) is placed the first time the IP falls to the procurement reorder level δ_P after an elapsed time of

$$\left(\frac{n - n_1 r}{r}\right) \frac{Q_R}{\bar{X}}$$

since the arrival of a procurement order, where

$$n_1 = \frac{T_R}{\tau_R} - \left[\frac{T_R}{\tau_R} \right] .$$

The [Z] is the greatest integer in Z. The fixed repair orders (Q_R) are made the first time the IP falls to δ_R after the arrival of a repair induction. In order to simplify the problem somewhat, it is assumed that as a demand is made on RFI inventory, a carcass is returned to NRFI inventory of which r% are reparable. With this assumption, it is possible to determine the exact amount of NRFI inventory, given that we know the amount of RFI inventory. Since the system to be formulated is one in which there is one procurement per cycle and n repairs per cycle, the random cycle time (T) can be expressed in terms of the stochastic demand (X) and the quantity demanded per cycle,

$$T = \frac{Q_P + n Q_R}{X} . \tag{1}$$

In the NRFI inventory, there will be exactly rXT items that enter the inventory each cycle and there will be nQ_R items leaving. Hence, in steady state

$$r \bar{X} \bar{T} \doteq n Q_R$$

or using equation (1) and solving for Q_P yields

$$Q_P \doteq \frac{n(1 - r)\, Q_R}{r} \quad . \tag{2}$$

It is postulated that, on the average, at the time of the arrival of procurement orders and repair batches the net inventory will be at fixed positive buffer levels b_1 and b_2, respectively. The actual net inventory at times of order arrivals will, of course, fluctuate and, hence, the buffer levels are indicative of protection against stockouts.

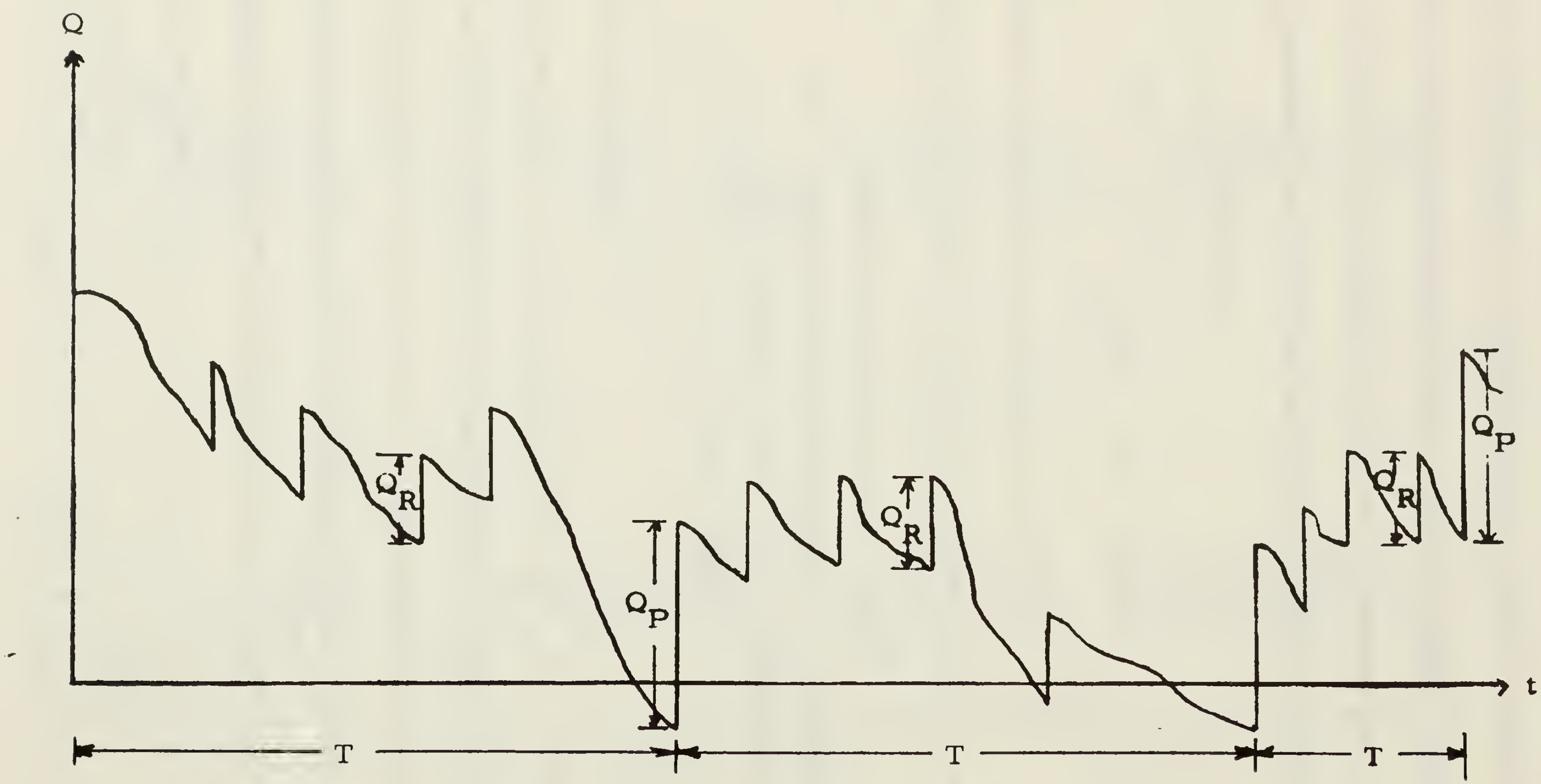
Every time an order is placed for either a procurement or a repair induction there is associated a cost assumed to be independent of the number of items either repaired or procured. These costs are those necessary to support the personnel and equipment involved in placing orders. The cost (C) of the items themselves is considered to be independent of the quantity procured. The RFI holding cost (h_1) is assumed to be equal to the cost of an item (C) times an inventory carrying charge (I), i.e.,

$$h_1 = IC \text{ in dollars per unit year .}$$

Since the NRFI inventory is made up of just carcasses, the holding cost (h_2) is considerably less than h_1. It is assumed that h_2 is equal to the cost of an item (C), minus the cost of repair (C_R), all times the inventory carrying charge (I), i.e.,

$$h_2 = I(C - C_R), \text{ in dollars per unit year .}$$

The shortage cost, or cost of incurring back orders, is assumed to be a function of the number of items backordered. The cost per back order (Π) is an intangible type of cost and in most cases should be assigned by the decision-maker. The value of Π has a direct effect on the desirability of incurring back orders; i.e., large values of Π make back orders extremely undesirable. The value assigned to Π is, in most cases, much greater than the value associated with holding either RFI or NRFI inventory. This being the case, the encountering of back orders will be the exception rather than the rule. Having made this assumption, an expected value formulation, using the expected values as parameters, can be used to determine the cost of holding both RFI and NRFI inventory; but an exact procedure will have to be employed to determine the expected number of back orders. Figure 1 gives an indication as to how the system really behaves; and Figures 2, 3, and 4 depict the expected value formulation.



Stochastic Demand Representation of the RFI Inventory
Depicting Only the Net Inventory

FIGURE 1

3.2.3 The Model

3.2.3.1 Introduction

With the preceding assumptions in hand, an expression will be developed for the expected cost of operating the system per cycle. This cost, divided by the cycle time (T), will yield an expression for the expected cost of operating the system per unit time in terms of known constants and the decision variables Q_P, Q_R, δ_R, δ_P, and n. The cost per cycle is given by:

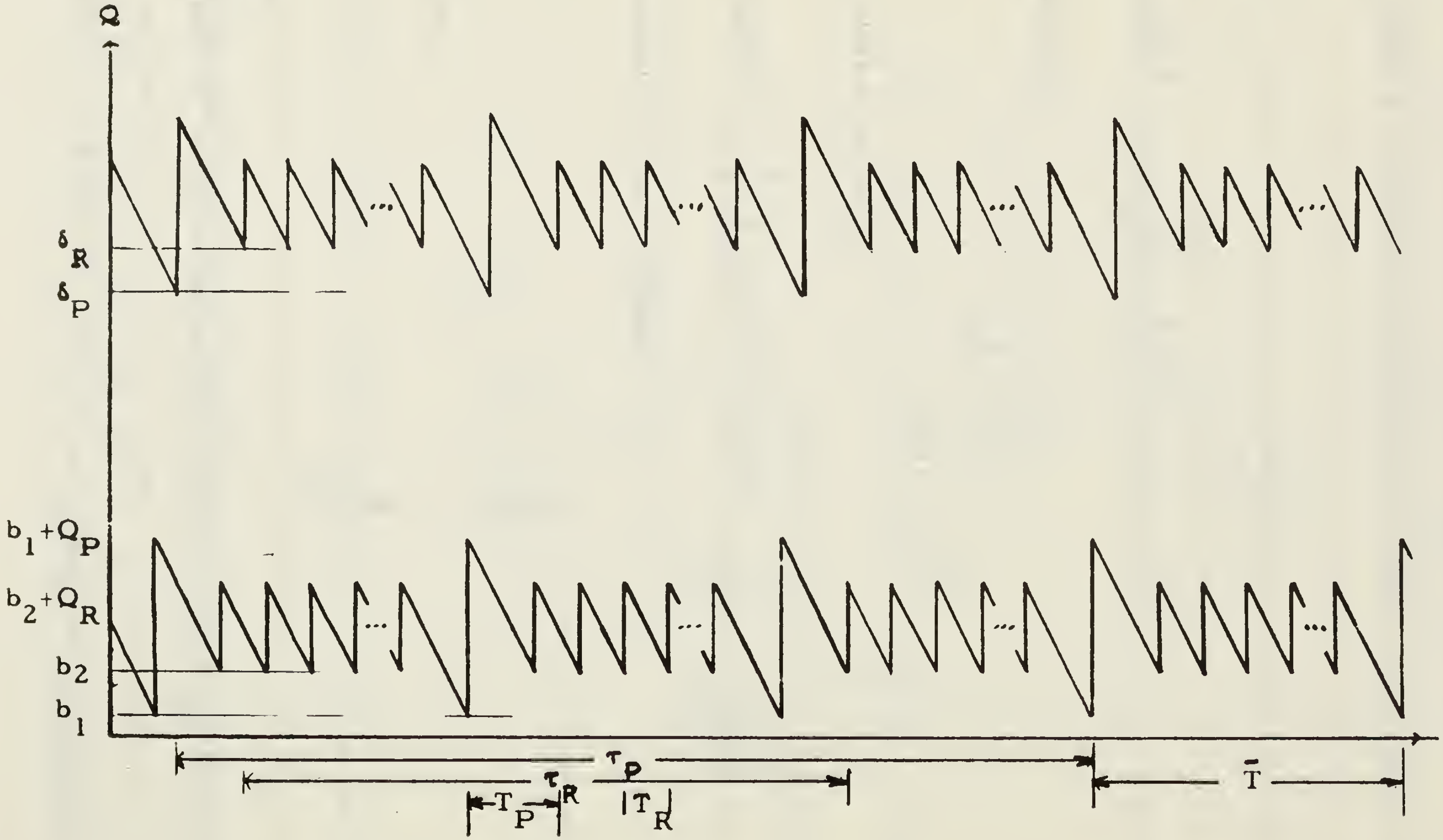
$$\begin{aligned} TC_T = \; & \text{order cost per cycle } (ORD_T) \; + \\ & \text{RFI holding cost per cycle } (RFI\,HOL_T) \; + \\ & \text{NRFI holding cost per cycle } (NRFI\,HOL_T) \; + \\ & \text{shortage cost per cycle } (SHG_T) \; . \end{aligned}$$

The order cost will simply be the cost of placing one procurement order plus the cost of having n inductions from repair, i.e.,

$$ORD_T = A_P + nA_R \; . \tag{3}$$

3.2.3.2 RFI Holding Cost

The expected holding cost per cycle for RFI inventory will be given by the product of the holding cost per unit, per unit time (h_1), and the area under the RFI inventory curve during the cycle. The area under the curve, see Figure 2, will be the area of the rectangle with sides b_1 and T_P, the area of the rectangle with sides b_2 and nT_R, the area of the triangle with base T_P and height Q_P, and the area of n triangles with base T_R and height Q_R.



Expected Value Representation of the RFI Inventory
Depicting Both Net Inventory and Inventory Position

FIGURE 2

$$\text{RFI HOL}_T = h_1 \left\{ b_1 T_P + nb_2 T_R + \frac{Q_P T_P}{2} + \frac{n Q_R T_R}{2} \right\} . \quad (4)$$

The time T_P is the expected time it takes for Q_P items to be demanded; hence,

$$T_P = \frac{Q_P}{\bar{X}}$$

and (5)

$$T_R = \frac{Q_R}{\bar{X}} \quad .$$

Substituting (5) into (4) yields

$$\text{RFI HOL}_T = \frac{h_1}{\bar{X}} \left\{ b_1 Q_P + nb_2 Q_R + \frac{Q_P^2}{2} + \frac{nQ_R^2}{2} \right\} . \quad (6)$$

Now using equation (2),

$$Q_P \doteq \frac{n(1 - r) Q_R}{r} \quad ,$$

(4) becomes

$$\text{RFI HOL}_T = \frac{h_1 Q_R n}{r \bar{X}} \left\{ b_1 (1 - r) + r b_2 + \frac{Q_R}{2r} \left[r^2 + n(1 - r)^2 \right] \right\} . \quad (7)$$

The buffer b_1 can now be determined by again referring to Figure 2 and noting the inventory position just before a procurement order is placed and what happens to the system during the procurement leadtime τ_P . The inventory position is at the trigger level δ_P . During the procurement leadtime τ_P , all the procurement orders previously placed will have arrived and all the outstanding orders for repair inductions will also have arrived. Some nK (K is to be determined) repair inductions whose orders have not yet been placed will also arrive. The quantity that leaves the system during this lead time will be the mean procurement leadtime demand $\bar{X}\tau_P = Z_P$. Hence, the expected buffer will be

$$b_1 = \delta_P - Z_P + nK\,Q_R \quad . \tag{8}$$

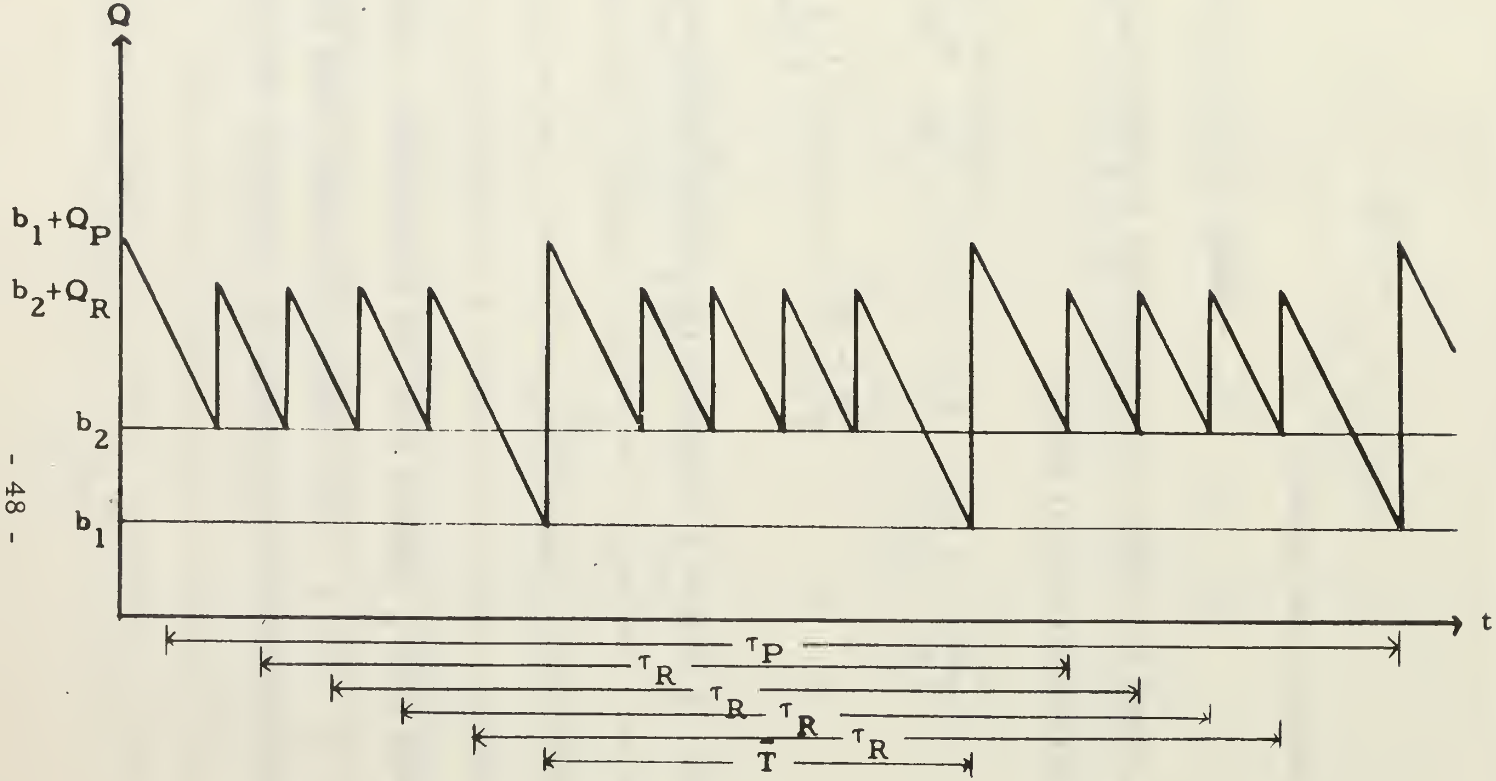
By similar analysis, the buffer b_2 is

$$b_2 = \delta_R - Z_R - K\,Q_P \quad , \tag{9}$$

where KQ_P items previously ordered will not arrive during τ_R. Substituting (8) and (9) into (7) yields

$$\text{RFI HOL}_T = \frac{h_1 Q_R n}{r\bar{X}} \left\{ (\delta_P - Z_P)(1 - r) + (\delta_R - Z_R) r + \frac{Q_R}{2r}\left[r^2 + n(1 - r)^2 \right] \right\} \quad . \tag{10}$$

In general, K will be a function of the number of cycles of random length T that occur during the lead times. Figure 3, for example, indicates that there are slightly less than three cycles occurring during the repair lead time. Here it is obvious K is equal to one. If τ_P remains the same and τ_R happens to be one cycle shorter, then in this example $K = 2 - 0 = 2$, which implies that eight repair inductions are ordered and arrive during the procurement lead time. For the general case,



Expected Value Representation of RFI Inventory
Depicting Relationships Between $\bar{T}$, τ_R, and τ_P

FIGURE 3

$$K = \left[\frac{\tau_P}{\bar{T}}\right] - \left[\frac{\tau_R}{\bar{T}}\right] = \left[\frac{Z_P\, r}{n Q_R}\right] - \left[\frac{Z_R\, r}{n Q_R}\right] , \tag{11}$$

where $[Z]$ indicates the greatest integer in Z .

3.2.3.3 NRFI Holding Cost

The NRFI holding cost will be obtained in a manner similar to that of the RFI holding cost. In this case, the area in question is depicted in Figure 4. This area will be the areas of the rectangle with sides b_3 and T, the triangle with base T and height nQ_R, and $\frac{n(n-1)}{2}$ parallelograms of horizontal length T_R and height Q_R. The NRFI holding cost will then be

$$\text{NRFI HOL}_T = h_2 \left\{ b_2 \bar{T} + \frac{n}{2} Q_R \bar{T} - \frac{n(n-1)}{2\bar{X}} Q_R^{\,2} \right\} . \tag{12}$$

During the repair lead time, the NRFI inventory must supply the RFI inventory with $nQ_R \left(\frac{\tau_R}{\bar{T}}\right)$ items. In steady state, the level of the buffer b_3 is the lowest possible level of the NRFI inventory, i.e.,

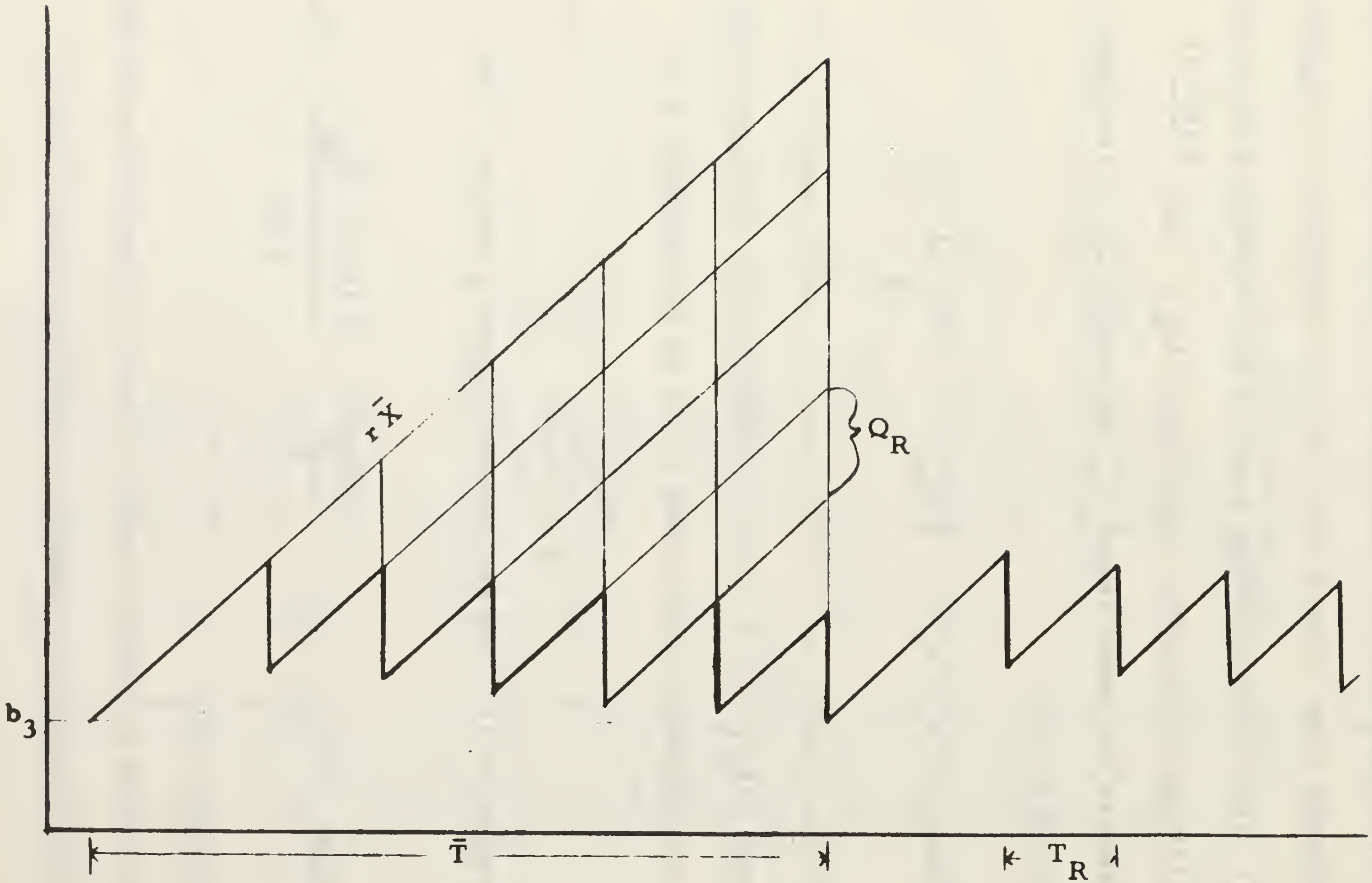
$$b_3 = nQ_R \left(\frac{\tau_R}{\bar{T}}\right) = r Z_R . \tag{13}$$

Substituting equation (13) into (12) the NRFI holding cost per cycle becomes

$$\text{NRFI HOL}_T = h_2 \left\{ r Z_R \bar{T} + \frac{nQ_R \bar{T}}{2} - \frac{n(n-1)\, Q_R^{\,2}}{2\bar{X}} \right\} . \tag{14}$$

3.2.3.4 Shortage Cost

Departing from an expected value analysis, an expression for the shortage cost per cycle is obtained by examining Figures 1 and 2 .



Expected Value Representation of the NRFI Inventory

FIGURE 4

Here back orders can be incurred at $(n + 1)$ different times during the cycle. There is a possibility of stock-out just before the procurement quantity Q_P arrives and just before the arrival of each of the n repair batches. The expected number of shortages, E, per cycle can be obtained by examining what takes place during the lead times. Just before a repair order is placed, the IP is at a level δ_R. As discussed previously, at the end of the repair lead time all the items previously ordered will have arrived with the exception of KQ_P. The number of items short will be zero as long as the leadtime demand is less than $\delta_R - KQ_P$; hence, the expected number of items short of the end of n repair lead times is

$$n \int_{\delta_R - KQ_P}^{\infty} (x - \delta_R + KQ_P)\, f(x;\tau_R)\, dx \quad .$$

Using a similar analysis on the procurement lead time, the expected number of items short of the end of this period is

$$\int_{\delta_P + nKQ_P}^{\infty} (x - \delta_P - nKQ_P)\, f(x;\tau_P)\, dx \quad ,$$

where there are nKQ_R items that are ordered and come in during the lead time. The expected number of items short per cycle will be

$$E = \int_{\delta_P + nKQ_R}^{\infty} (x - \delta_P - nKQ_R)\, f(x;\tau_P)\, dx$$
$$+ n \int_{\delta_R - KQ_P}^{\infty} (x - \delta_R + KQ_P)\, f(x;\tau_R)\, dx \quad , \qquad (15)$$

where x is a dummy variable indicating leadtime demand and K is given by equation (11). The expected cost of back orders per cycle is simply

$$SHC_T = \Pi E \quad . \tag{16}$$

3.2.3.5 Total Cost

The total cost per cycle is given by the expression

$$TC_T = ORD_T + RFI\,HOL_T + NRFI\,HOL_T + SHC_T \quad .$$

Since

$$T = \frac{(Q_P + nQ_R)}{X} \doteq \frac{nQ_R}{r\bar{X}} \quad ,$$

the total cost per year will be

$$TC = \frac{r\bar{X}}{nQ_R} \left\{ ORD_T + RFI\,HOL_T + NRFI\,HOL_T + SHC_T \right\} \quad . \tag{17}$$

Substituting equations (3), (10), (14), (15), and (16) into (17), the total cost per year becomes

$$TC = \frac{(A_P + nA_R)}{nQ_R} \bar{X} r + h_1 \Big\{ (\delta_P - Z_P)(1 - r) + (\delta_R - Z_R) r$$

$$+ \frac{Q_R}{2r} \left[r^2 + n(1 - r)^2 \right] \Big\}$$

$$+ h_2 \left\{ r Z_R + \frac{Q_R}{2} \left[n - r(n - 1) \right] \right\}$$

$$+ \frac{\Pi \bar{X} r}{n Q_R} \Big\{ \int_{\delta_P + nKQ_R}^{\infty} (x - \delta_P - nKQ_R) \; f(x; \tau_P) \, dx$$

$$+ n \int_{\delta_R - \frac{n(1 - r)}{r} KQ_R}^{\infty} \left(x - \delta_R + \frac{n(1 - r)}{r} KQ_R \right)$$

$$\cdot f(x; \tau_R) \, dx \Big\} . \qquad (18)$$

Before attempting to minimize (18) with respect to the decision variables, a value for K must be determined. From equation (11), K is given as

$$K = \left[\frac{Z_P r}{nQ_R} \right] - \left[\frac{Z_R r}{nQ_R} \right]$$

which can be rewritten as

$$K = \left(\frac{Z_P r}{nQ_R} - D_1 \right) - \left(\frac{Z_R r}{nQ_R} - D_2 \right)$$

or

$$K = \frac{(Z_P - Z_R) r}{nQ_R} - (D_1 - D_2) \qquad (19)$$

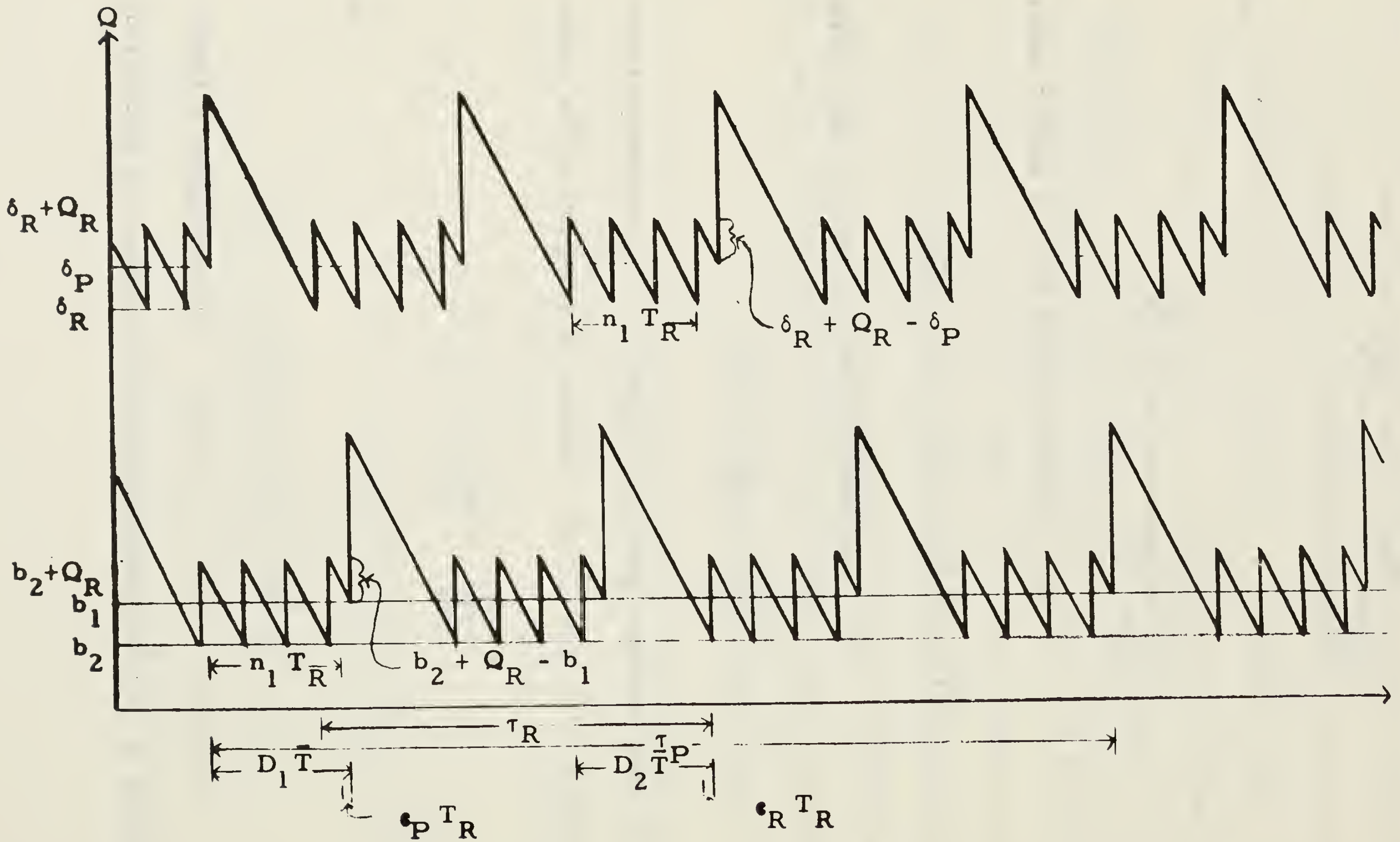
where

$$0 \leq D_i < 1 \,, \quad \text{for} \quad i = 1, 2 \,.$$

Substituting (19) into (18), the total cost per year becomes

$$\begin{aligned}
TC = {} & \frac{(A_P + n A_R)}{n Q_R} \, r \bar{X} + h_1 \Bigg\{ (\delta_P - Z_P)(1 - r) + (\delta_R - Z_R) r \\
& + \frac{Q_R}{2r} \left[r^2 + n (1 - r)^2 \right] \Bigg\} \\
& + h_2 \left\{ r Z_R + \frac{Q_R}{2} \left[n - r (n - 1) \right] \right. \\
& + \frac{\Pi r \bar{X}}{n Q_R} \left\{ \int_{\delta_R + r (Z_P - Z_R) - n Q_R (D_1 - D_2)}^{\infty} (x - \delta_P - r (Z_P - Z_R) + n Q_R (D_1 - D_2)) \right. \\
& \qquad \cdot f(x ; \tau_P) \, dx \\
& + n \int_{\delta_R - (1 - r)(Z_P - Z_R) + \frac{n (1 - r)}{r} Q_R (D_1 - D_2)}^{\infty} (x - \delta_R + (1 - r)(Z_P - Z_R) - \frac{n (1 - r)}{r} \\
& \qquad \left. Q_R (D_1 - D_2)) \, f(x ; \tau_R) \, dx \right\} .
\end{aligned} \tag{20}$$

Referring to Figure 5, a more exact analysis can be made on the values of D_1 and D_2. It is noted that D_1 is the time between placing of a procurement order and the arrival of the next procurement, which had previously been ordered. Noting the net inventory, it is apparent



Expected Value Representation of RFI Inventory
Depicting Relationships Between D_1 and D_2

FIGURE 5

that during this same amount of time some n_1, ($n_1 \leq n$), inductions arrive from the NRFI inventory. Hence, D_1 can be written as

$$D_1 \bar{T} = n_1 T_R + \epsilon_P T_R$$

or

$$D_1 = \frac{n_1 r}{n} + \frac{\epsilon_P r}{n} \quad ,$$

where ϵ_P is that fraction between $n_1 T_R$ and $(n_1 + 1) T_R$ which is also included in D_1. A similar analysis indicates that D_2 is given by

$$D_2 = \frac{n_1 r}{n} + \frac{\epsilon_R r}{n} \quad .$$

Hence, the fraction D can be written as

$$D = D_1 - D_2 = \frac{r}{n} (\epsilon_P - \epsilon_R) \quad .$$

From Figure 5, it is readily apparent that

$$\epsilon_P T_R < \frac{b_2 + Q_R - b_1}{\bar{X}}$$

or

$$\epsilon_P < \frac{b_2 - b_1 + Q_R}{Q_R} \quad .$$

Also,

$$\epsilon_R < \frac{\delta_R + Q_R - \delta_P}{Q_R} \quad .$$

Since ϵ_P and ϵ_R are both positive, then

$$\epsilon_P - \epsilon_R < \left| \frac{b_2 - b_1 - \delta_R + \delta_P}{Q_R} \right|$$

where $|Z|$ is the absolute value of Z. But,

$$b_1 - b_2 \approx \delta_P - \delta_R \ .$$

Consequently,

$$\epsilon_P - \epsilon_R \approx 0 \ .$$

3.2.3.6 Determination of Decision Variables

Now, in order to operate the system economically, values of the decision variables must be obtained which minimize the total cost per unit time. These values will be obtained by setting the first order conditions equal to zero. This will be done for the decision variables Q_R, δ_P, and δ_R only; the integer n will be determined by a stepping process, and Q_P follows Q_R through equation (2). This stepping process will start by setting $n = 1$, for which a value of Q_R, δ_R, δ_P, and TC will be computed. The value of n will be increased by one, and the procedure repeated. The process will continue until the minimum cost is obtained. The first order conditions are obtained by taking the partial derivatives of the total cost equation (20) with respect to the decision variables Q_R, δ_P, and δ_R. First the partial derivative of total cost per year with respect to Q_R is

$$\frac{\partial TC}{\partial Q_R} = - \frac{(A_P + n A_R)}{n Q_R^2} r \bar{X} + \frac{h_1}{2r} \left\{ r^2 + n(1-r)^2 \right\}$$

$$+ \frac{h_2}{2} \left\{ n - r(n-1) \right\}$$

$$- \frac{\Pi r \bar{X}}{n Q_R^2} \left\{ \int_{\delta_P + r(Z_P - Z_R) - nQ_R(D_1 - D_2)}^{\infty} (x - \delta_P - r(Z_P - Z_R))\, f(x; \tau_P)\, dx \right.$$

$$\left. + n \int_{\delta_R - (1-r)(Z_P - Z_R) + \frac{n(1-r)}{r} Q_R (D_1 - D_2)}^{\infty} (x - \delta_R + (1-r)(Z_P - Z_R))\, f(x; \tau_R)\, dx \right\} . \tag{21}$$

It becomes quickly apparent that equation (21) cannot be solved explicitly for either Q_R, δ_P, or δ_R ; but an expression can be obtained for Q_R in terms of Q_R, δ_P, and δ_R. Setting equation (21) equal to zero, it may be written as

$$Q_R = \left[\left\{ \frac{\frac{2 r^2 \bar{X}}{n}}{h_1 (r^2 + n(1-r)^2) + h_2 (rn - r^2 (n-1))} \right\} \left\{ A_P + n A_R \right. \right.$$

$$+ \Pi \left(\int_{\delta_P + r(Z_P - Z_R) - nQ_R (D_1 - D_2)}^{\infty} (x - \delta_P - r(Z_P - Z_R)) \, f(x; \tau_P) \, dx \right.$$

$$+ n \int_{\delta_R - (1-r)(Z_P - Z_R) + \frac{n(1-r)}{r} Q_R (D_1 - D_2)}^{\infty} (x - \delta_R + (1-r)(Z_P - Z_R))$$

$$\left. \left. \left. \cdot \, f(x; \tau_R) \, dx \right) \right\} \right]^{\frac{1}{2}} . \quad (22)$$

Next, the partial derivative of total cost per unit time with respect to δ_R is

$$\frac{\partial TC}{\partial \delta_R} = h_1 r - \frac{\Pi r \bar{X}}{Q_R} \Big[1 - F(\delta_R - (1-r)(Z_P - Z_R)$$

$$+ \frac{n(1-r)}{r} Q_R (D_1 - D_2); \tau_R) \Big] \quad (23)$$

where $F(x;t)$ is the cumulative distribution function of demand during time t. Setting (22) equal to zero, it can be written as

$$\frac{h_1 Q_R}{\Pi \bar{X}} = \Big[1 - F \big(\delta_R - (1 - r) (Z_P - Z_R) + \frac{n (1 - r)}{r} Q_R (D_1 - D_2) ; \tau_R \big) \Big] . \tag{24}$$

Finally, the partial derivative of total cost per unit time with respect to δ_P is

$$\frac{\partial TC}{\partial \delta_P} = h_1 (1 - r) - \frac{\Pi r \bar{X}}{n Q_R} \Big[1 - F \big(\delta_P + r (Z_P - Z_R) - n Q_R (D_1 - D_2) ; \tau_P \big) \Big] . \tag{25}$$

Here again there is no way of expressing any of the decision variables explicitly in terms of the other, but an expression can be obtained which will yield a solution through iterations with equations (22) and (24). Setting (25) equal to zero, it may be written as

$$\frac{h_1 Q_R n (1 - r)}{\Pi r \bar{X}} = \Big[1 - F \big(\delta_P + r (Z_P - Z_R) - n Q_R (D_1 - D_2) ; \tau_P \big) \Big] . \tag{26}$$

If the decision variables Q_P had been used instead of Q_R, equations (22), (24), and (26) would have been

$$Q_P = \Bigg[\left\{ \frac{2(1-r)^2 n\bar{X}}{h_1(r^2 + n(1-r)^2) + h_2(rn - r^2(n-1))} \right\} \Bigg\{ A_P + nA_R$$

$$+ \Pi \left(\int_{\delta_R + r(Z_P - Z_R) - \frac{r}{1-r} Q_P (D_1 - D_2)}^{\infty} (x - \delta_P - r(Z_P - Z_R))\, f(x; \tau_P)\, dx \right. \tag{27}$$

$$\left. \left. + n \left(\int_{\delta_P - (1-r)(Z_P - Z_R) + Q_P(D_1 - D_2)}^{\infty} (x - \delta_P + (1-r)(Z_P - Z_R))\, f(x; \tau_R)\, dx \right) \right\} \right]^{\frac{1}{2}}$$

and

$$\frac{h_1 r Q_P}{n(1-r)\Pi \bar{X}} = \Big[1 - F(\delta_R - (1-r)(Z_P - Z_R) + Q_P(D_1 - D_2); \tau_R) \Big] \tag{28}$$

and

$$\frac{h_1 Q_P}{\Pi \bar{X}} = \Big[1 - F(\delta_P + r(Z_P - Z_R) - \frac{r}{1-r} Q_P (D_1 - D_2); \tau_P) \Big]. \tag{29}$$

As a sidelight, it might be of interest to note that if $r = 0$, above equations would become

$$Q_P = \left[\frac{2\bar{X}}{h_1} \left(A_P + \Pi \int_{\delta_P}^{\infty} (x - \delta_P)\, f(x; \tau_P)\, dx\right) \right]^{1/2}$$

and

$$\frac{h_1 Q_P}{\Pi \bar{X}} = \left[1 - F(\delta_P; \tau_P) \right]$$

which are identical to the results obtained in consumable inventory theory [20] using a similar approach.

3.2.3.7 Procedures to Obtain Solutions

There are several methods of obtaining solutions to the above equations. One which makes no assumptions on the fraction D is as follows:

1. Set $n = 1$.
2. Solve for an initial Q_R using the equation

$$Q_R = \frac{r}{n(1-r)} \sqrt{\frac{2 A_P \bar{X}}{h_1}} .$$

3. Solve for $D = (D_1 - D_2)$ using the Q_R from step 2 as follows:

$$D = \frac{(Z_P - Z_R)\, r}{n Q_R} - \left(\left[\frac{Z_P\, r}{n Q_R} \right] - \left[\frac{Z_R\, r}{n Q_R} \right] \right)$$

   where $[Z]$ is the greatest integer in Z.
4. Substitute this value of D along with the values of Q_R and n into equations (24) and (26) to determine δ_R and δ_P.

5. Using the determined values of Q_R, δ_P, δ_R, D, and n in the right-hand side of (22), determine a new Q_R.

6. Continue steps 3, 4, and 5 until the three equations converge. If they do not converge, assume a value for D and repeat steps 3 and 5 until they do. Since the value of D is very small, a good value to assume would be zero.

7. Compute the total cost per unit time with the resulting values of Q_R, δ_R, δ_P, n, and K.

8. Increase n by one and repeat steps 2 through 8 until the minimum cost is determined. Select those values of Q_R, δ_R, δ_P, and n which give the minimum cost and compute Q_P from equation (2).

A second procedure would be to set $D = 0$ and repeat procedure one, keeping D at zero. This procedure should be used only in the event that the results of procedure one prove to be infeasible.

3.2.4 A Numerical Example

Since this paper is an extension of the substitution policy, the following example is the same as that presented in reference [1]. The values of the parameters were given as follows: $A_P = \$750$, $A_R = \$100$, $r = 0.9$, $\bar{X} = 1{,}000$ units per year, $h_1 = \$200$ per unit year, $h_2 = \$20$ per unit year, $\tau_P = 1.0$ years, $\tau_R = 0.25$ years. The value of h_1 was based on a unit cost (C) of \$1,000 per unit, which implies a carrying charge (I) of 0.2 per year, and the cost of repair of \$900 per unit. Here a shortage cost per unit (Π) is set equal to \$1,000, and the demand on the RFI inventory is assumed to have a normal distribution with mean and variance both equal to $\bar{X}t$. The first procedure set forth at the end of the previous section was followed but the results, listed in Table 1, obtained for most of the steps of the integer n yielded values of the fraction D that were outside the permissible range set forth in the previous section. The value of the fraction was set equal to zero, and the procedure was repeated keeping D at zero. The results, listed in Table 2, of this second procedure were almost identical to those obtained in the first except that the reorder points changed considerably. With the fraction outside its permissible range, the values obtained for the reorder points were impossible; i.e., once the IP got down to the δ_R level, it would be impossible for it to get back up to the δ_P level. For example, the trigger levels obtained for $n = 17$ were $\delta_R = 356$, $\delta_P = 465$, and the repair quantity was 36 units. Once the IP got down to $\delta_R = 356$, the highest it could ever be after this would be $\delta_R + Q_R = 392$. Since $\delta_R + Q_R$ is less than δ_P, a procurement order would never again be placed and the net inventory (NI) would tend toward negative infinity. The results for $D = 0$, with rounding off, were: 17 lot-size inductions from the repair facility of 35 units each; a procurement quantity of 67 units; an average cycle length of 0.67 years; reorder points, based on IP, of $\delta_R = 363$ and $\delta_P = 394$; all at an expected total cost per year of $TC = \$18,831.67$. It must be mentioned that this total cost does not include the cost of items nor the cost of repairing items. The results obtained are in excellent agreement with those in the example of Section 2, which are: 19 inductions of 30 units each; a procurement of 63 items; and a cycle (deterministic) time of 0.63 years. An examination of Tables 1 and 2 indicates that the buffers b_1 and b_2 are approximately the same for both procedures. With the buffers the same, it is expected that the cost should be the same; consequently, by setting the fraction D equal to zero, the order quantities and costs do not change but trigger levels that are feasible, are obtained. Again, an examination of the values in Tables 1 and 2 quickly reveals that the total cost of operating the system is quite insensitive to the order quantities and the number of inductions in the neighborhood of the optimum. These results are consistent with those obtained in consumable inventory theory [20] in that a plot of total cost versus procurement quantity is quite flat in the neighborhood of the optimum.

Even as complicated as the equations appear, they are quite easily solved with the aid of a computer. The previous example was programed in FORTRAN 60 and run on a CDC 1604, with a total run time of slightly less than two minutes. The results of the iterations for $n = 13$ and $D = 0$ are listed in Table 3. As can be seen from this table, the equations converge quite rapidly, which was the case for all steps of n from one through 25.

In order to get some indication as to how the repair leadtime affects the total operating cost of the system per year, the above example was run again on the computer with τ_R set equal to 0.5 years vice 0.25 years. The results of this run are listed in Table 4. As can be seen by comparing Tables 2 and 4, the total cost of maintaining the system can be reduced for this example by almost $8,000 per year by decreasing the repair leadtime from 0.5 years to 0.25 years.

n	Q_R	Q_P	δ_R	δ_P	b_1	b_2	D	TC	T
1	93	10	355	438	91	32	0.24	22897.08	0.103
2	70	16	362	382	86	34	-0.20	20943.00	0.156
3	61	20	346	538	87	35	0.70	20156.25	0.201
4	55	24	359	423	83	36	0.07	19727.19	0.244
5	50	28	372	316	79	37	-0.35	19459.28	0.283
6	48	32	383	212	77	37	-0.66	19278.91	0.321
7	46	36	358	435	77	37	0.10	19151.79	0.357
8	44	39	365	370	73	37	-0.08	19059.61	0.392
9	43	43	372	308	75	37	-0.24	18991.66	0.426
10	41	46	380	247	75	38	-0.37	18941.25	0.459
11	40	49	386	188	75	38	-0.48	18903.92	0.492
12	39	52	340	601	75	37	0.43	18876.64	0.523
13	38	55	343	573	74	37	0.35	18857.23	0.554
14	38	58	346	545	74	37	0.28	18844.08	0.585
15	37	61	349	518	72	37	0.22	18836.02	0.614
16	36	64	353	491	68	39	0.17	18832.12	0.643
17	36	67	356	465	67	39	0.12	18831.67	0.672
18	35	70	358	439	70	38	0.07	18834.12	0.700
19	34	73	361	414	69	38	0.03	18839.00	0.728
20	34	75	364	389	70	38	-0.01	18845.95	0.755
21	33	78	367	365	68	39	-0.04	18854.68	0.781
22	33	81	370	341	67	38	-0.07	18864.93	0.805
23	33	83	372	317	67	39	-0.10	18876.50	0.833
24	32	86	375	294	69	39	-0.13	18889.21	0.859
25	32	88	378	271	65	40	-0.15	18902.92	0.884

Results of Numerical Example Using Procedure 1

TABLE 1

n	Q_R	Q_P	δ_R	δ_P	b_1	b_2	TC	T
1	93	10	357	415	90	32	22897.08	0.103
2	70	16	359	411	86	34	20943.00	0.156
3	61	20	360	408	83	35	20156.25	0.201
4	55	24	361	406	81	36	19727.19	0.244
5	51	28	362	405	80	37	19459.28	0.283
6	48	32	362	403	78	37	19278.92	0.321
7	46	36	362	402	77	37	19151.79	0.357
8	44	39	362	401	76	37	19059.61	0.392
9	43	43	363	400	75	38	18991.66	0.426
10	41	46	363	400	75	38	18941.25	0.459
11	40	49	363	399	74	38	18903.92	0.492
12	39	52	363	398	73	38	18876.64	0.523
13	38	55	363	397	72	38	18857.22	0.554
14	38	58	363	397	72	38	18844.08	0.585
15	37	61	364	396	71	39	18836.02	0.614
16	36	64	364	396	71	39	18832.12	0.643
17	36	67	364	395	70	39	18831.67	0.672
18	35	70	364	394	69	39	18834.12	0.700
19	34	73	364	394	69	39	18839.00	0.728
20	34	75	364	393	68	39	18845.95	0.754
21	33	78	364	393	68	39	18854.68	0.781
22	33	81	364	393	68	39	18864.93	0.807
23	33	83	364	392	67	39	18876.50	0.833
24	32	86	364	392	67	39	18889.21	0.859
25	32	88	364	392	67	39	18902.92	0.884

Results of Numerical Example Using Procedure 2

TABLE 2

n	Q_R	Q_P	δ_R	δ_P	TC
13	100	144	357	385	25479.71
13	46	67	362	395	19636.90
13	39	57	363	397	18957.23
13	39	56	363	397	18870.31
13	38	55	363	397	18858.94
13	38	55	363	397	18857.45
13	38	55	363	397	18857.26
13	38	55	363	397	18857.23
13	38	55	363	397	18857.23
13	38	55	363	397	18857.23
13	38	55	363	397	18857.23
13	38	55	363	397	18857.23

Representative Example of the Number of Iterations
Required to Determine Minimum Cost

TABLE 3

n	Q_R	Q_P	δ_R	δ_P	TC
1	96	11	596	640	30475
2	73	16	599	636	28627
3	63	21	600	633	27896
4	57	25	601	631	27503
5	53	29	602	630	27261
6	50	33	602	628	27101
7	48	37	602	627	26991
8	46	41	603	626	26913
9	45	45	603	625	26857
10	43	48	603	624	26818
11	42	51	603	623	26790
12	41	55	604	622	26772
13	40	58	604	622	26761
14	39	61	604	621	26755
15	39	64	604	621	26755
16	38	67	604	620	26757
17	37	70	604	619	26763
18	37	73	605	619	26772
19	36	76	605	618	26782
20	36	79	605	618	26795
21	35	82	605	618	26809
22	35	85	605	617	26824
23	34	87	605	617	26841
24	34	90	605	616	26858
25	33	92	605	616	26876

Results of Numerical Example Using Procedure 2
with the Repair Lead Time Equal to
One-Half Year Vice One-Fourth Year

TABLE 4

3.2.5 Conclusion

In addition to the basic assumptions listed previously, the model was developed with one particular type of reparable item in mind. This type of item can be classified as being essential with a high demand rate, which usually implies that back orders are highly undesirable. Items that are inexpensive, in general, do not meet the criteria of being reparable; hence, one more characteristic of the type of item in mind is that it is relatively expensive. Items that cannot be handled by the model are those which permit back orders to occur quite frequently. This class consists of those items which are quite expensive with a low demand rate. To handle this type of item, a model would have to be developed using an exact treatment of both the holding and shortage costs. Another type of item that could not be handled with the model would be one in which the distribution of demand changes considerably from one cycle to the next. An example of this type would be an item which was either being phased in or being phased out of the inventory system.

If the assumption dealing with the return of carcasses to NRFI is beyond all realm of possibility, then the repair lead time would have to be a random variable. This random variable would be a constant δ_R as long as there are enough items in NRFI inventory to accommodate a repair induction at the time a repair order is placed. If there are not enough items in NRFI inventory at the time an induction order is placed, then the lead time would be increased a random amount of time τ. This would be the time necessary to accumulate enough carcasses to satisfy the lot size restriction of Q_R items. In this case, the relationship between Q_P and Q_R would not exist, and the distribution of demand during the repair lead time would be a function of two random variables. Even with the above type of items excluded, there are still a considerable number of items that are of the category specified by the assumptions, and the model presented should result in values of the decision variables that are fairly accurate.

3.3 A Constrained Periodic Review Model of the Reparable Item Inventory System

3.3.1 Introduction and Notation

There are many ways of categorizing items in an inventory system. One way is to put all items into one of two areas, either reparable or non-reparable. In the military, due to the nature of its business, there is a large number of reparable-type items that represents a considerable dollar inventory. Realistically, although an item has been classified as a reparable type of item, not all of a particular type that wear out or fail can be repaired. For example, the AN/PRC-6 radio is classified as reparable, but not all the AN/PRC-6 radios that fail or become inoperative can be economically restored to an operating condition. Because of this, items must be procured from time to time to replenish the overall inventory system. Understanding this, one realizes that demands can be satisfied with items that are either procured or that have been returned to a repair facility and repaired.

Many models have been formulated for the consumable item inventory system. These models typically answer the questions of how much to procure and when to procure in order to minimize cost or shortages. However, when investigating a reparable item inventory system, not only the questions of how much and when to procure must be answered, but also the questions of how much and when to repair must be answered. Further, the reparable item inventory system should be viewed as a system, rather than separate inventories of procured and repaired ready-for-issue items.

The system discussed in this paper is reviewed periodically, and a procurement is made at the time of review. Demand, X, is a random variable with a density function $f(x, t)$ over the period t. The procurement lead time, τ_P, is a constant. Repaired items are returned to the ready-for-issue stock at fixed time intervals of η, and the quantity returned during a time t is a random variable, Y, with the density function $g(y, t)$. Further, all back orders will be filled.

If the criterion of minimizing total cost per unit time to operate the system were used, a shortage cost would have to be postulated.

This is extremely difficult to do when discussing military items. How many dollars does it cost the government or the people of the United States if a tank, jet fighter, or submarine is inoperable due to the lack of a spare part? Volumes of literature have been written attempting to answer this question, for example, Solomon, et al [21]. However, no one has yet provided a satisfactory method for assigning shortage costs for military equipment. For this reason, the chosen criteria is to minimize the number of units backordered, subject to an operating cost budget, and hence avoid postulating a shortage cost. This assumes that the problem of optimally segmenting an operating cost budget for an inventory control point into separate operating cost budgets for each type of item managed by the inventory control point is possible and has been accomplished.

This model is an approximate approach to the constrained periodic review system as described. The objective is to determine the optimal order up to level R for procurement, the optimal review period for procurement, T, and the optimal repair cycle time, η, such that the average number of back orders is minimized while maintaining the cost of operating the system less than or equal to the yearly budget.

The following notation is defined and summarized.

A_P	-	Cost to make one procurement.
A_R	-	Set-up cost per repair cycle.
B	-	Annual budget.
$f(x, t)$	-	Density function of the quantity of items, X, demanded in time t.
$g(y, t)$	-	Density function of the quantity of items, Y, repaired and returned to RFI_R in time t.
h_1	-	Holding cost for ready-for-issue items.
h_2	-	Holding cost for non-ready-for-issue items.
J	-	Cost to make one review.
K	-	Cost of operating the inventory system.
NRFI	-	Non-ready-for-issue.
R	-	Procurement "order up to" quantity.
RFI_P	-	That portion of ready-for-issue stock that was procured.
RFI_R	-	That portion of ready-for-issue stock that came from repair.
S	-	Expected number of units backordered per unit time.
T	-	Review cycle time, expressed in years.
V	-	Number of units backordered.
$\bar{x}$	-	Average annual demand rate.
$\bar{y}$	-	Average annual return rate of repaired items to RFI_R.
η	-	Fixed cycle time for return of reparable items to ready-for-issue stock.
Π	-	Lagrange multiplier.
τ_P	-	Procurement lead time, a given constant.

3.3.2 Model Formulation

3.3.2.1 General

The on-hand inventory of ready-for-issue (RFI) stock may be viewed as two separate inventories, one consisting of all procured RFI items and the other consisting of all repaired RFI items. We may consider that all demands during η are filled by the RFI stock of repaired items (RFI_R) until it is depleted. At that time, demands are then placed on the procured RFI stock (RFI_P) until the end of the time period η. At the beginning of the next time period η, we receive a variable quantity of items for RFI_R, and the process starts over. Figures 1 through 3 portray this process and the accumulation of the non-ready-for-issue (NRFI) stock (the notation will be introduced below).

In order to minimize the expected number of units backordered, subject to a budget constraint on operating cost, the total variable operating cost per unit time must be determined. The relevant components of the variable cost per unit time needed to be determined are:

(a) procurement order cost;

(b) review cost;

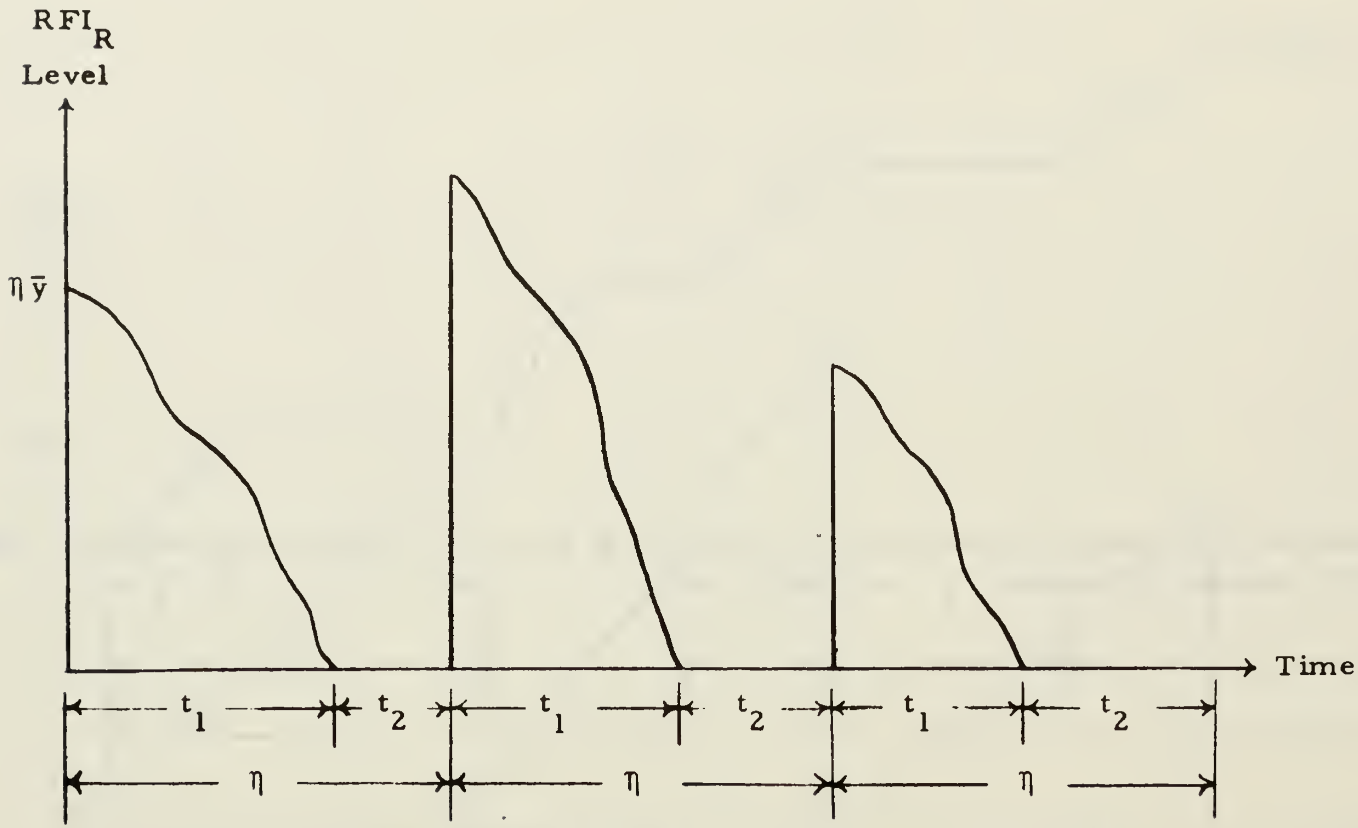
(c) repair set-up cost;

(d) NRFI holding cost;

(e) RFI_R holding cost;

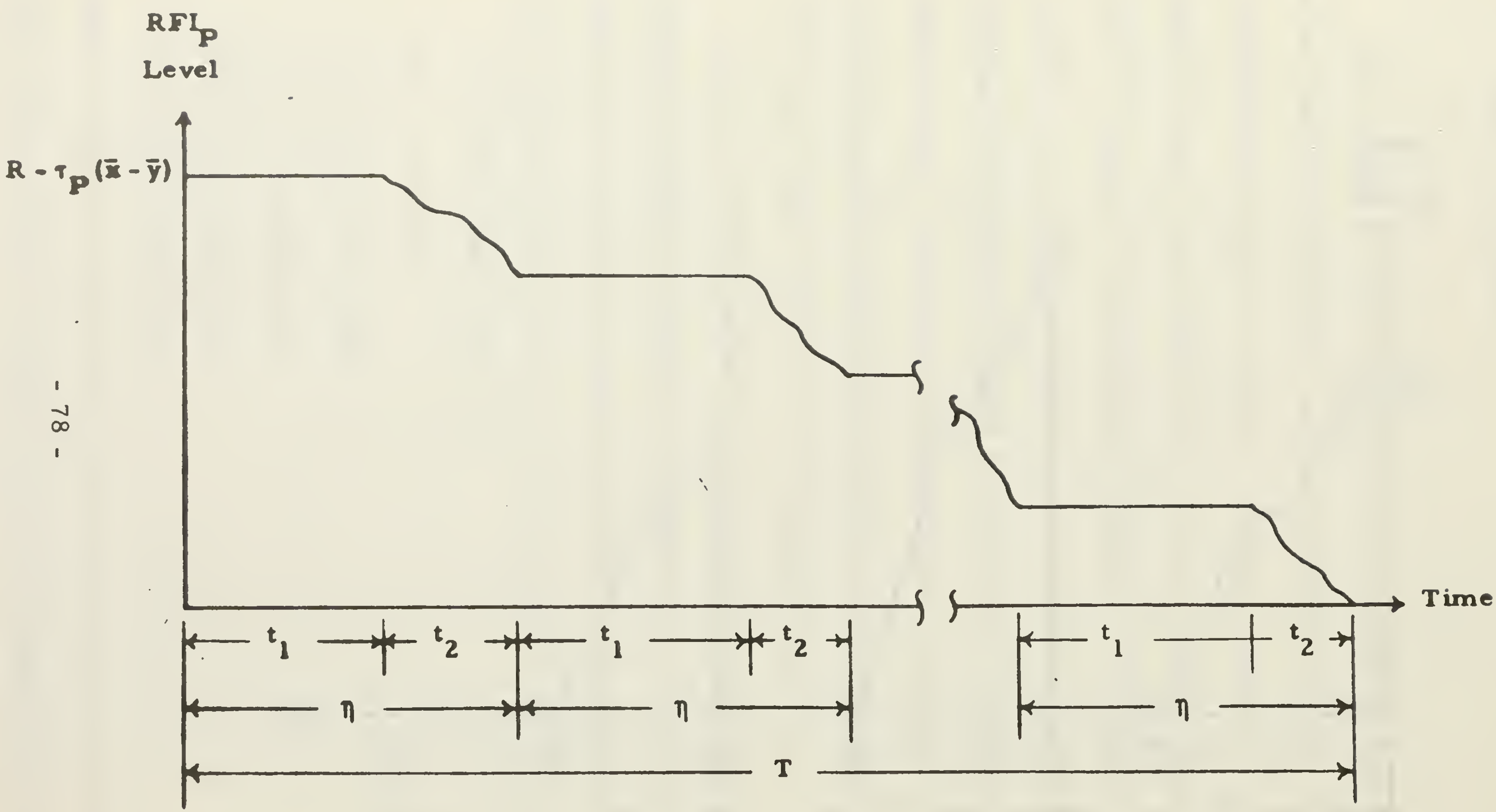
(f) RFI_P holding cost.

The following subsections develop these costs, the sum of which is the total variable cost per unit time. Then, the expression for determining



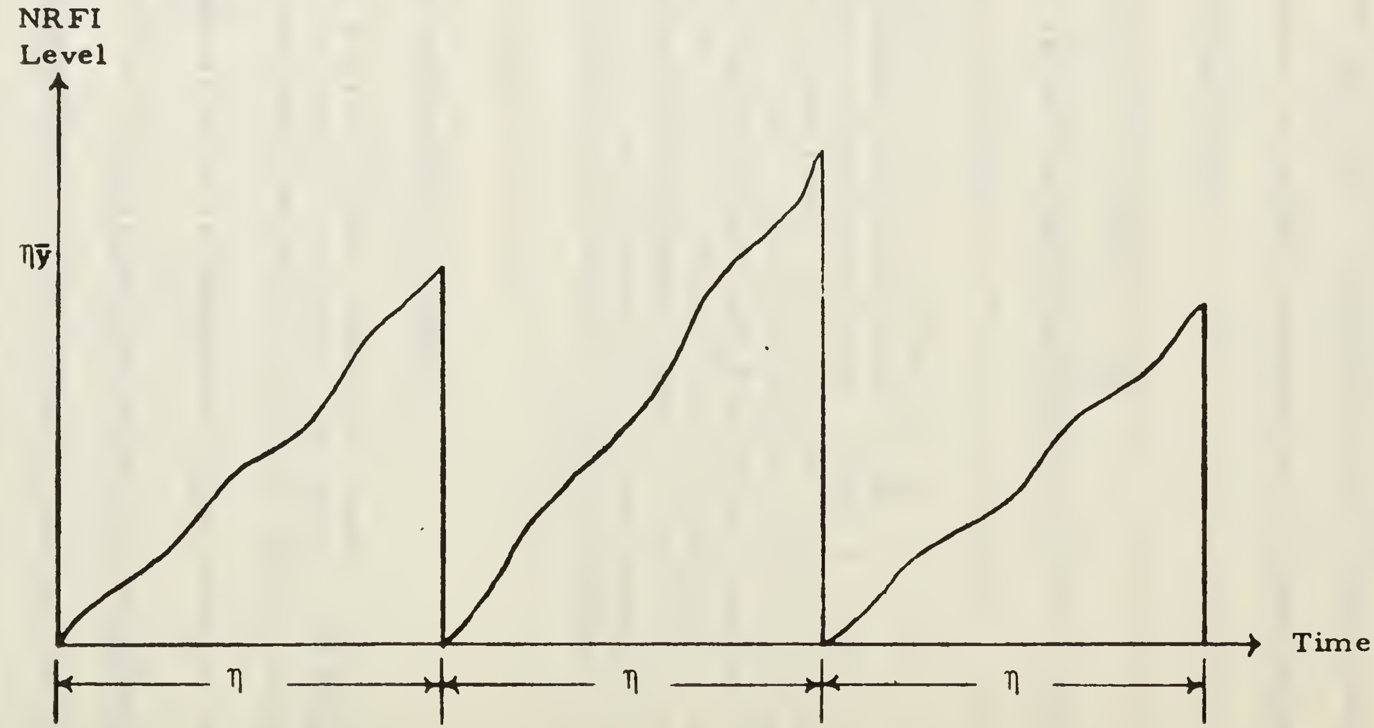
Repaired Ready-for-Issue Level Versus Time

FIGURE 1



Procured Ready-for-Issue Level Versus Time

FIGURE 2



Non-Ready-for-Issue Level Versus Time

the expected number of units backordered per unit time is developed. With this information, a solution to the problem may be determined. The formulation throughout is approximate, rather than exact. Inventory holding costs and set-up costs are determined by treating the expected values of random variables as parameters. The expected number of units short per unit time expression treats the random variables in the proper manner, but is only approximate for reasons which will be discussed when the expression is developed.

3.3.2.2 Order and Review Cost

In the description of the inventory system contained in the introduction, it was assumed that a procurement order was placed each time a review was made. The cost associated with placing one order is A_P . Since the review cycle is of fixed length, T, the order cost per unit time is $\frac{A_P}{T}$. Similarly, the review cost per unit time is $\frac{J}{T}$.

3.3.2.3 Repair Set-up Cost

The repair set-up cost per time period η is A_R . Therefore, the repair set-up cost per unit time is $\frac{A_R}{\eta}$.

3.3.2.4 NRFI Holding Cost

The dimensions of the NRFI holding cost, h_2, are dollars per unit year. Therefore, the unit years of stock held in NRFI inventory must be determined. Figure 3 portrays the NRFI inventory level over time. The average annual return rate of repaired items is $\bar{y}$. Since η is expressed in years, the expected height of each triangle is $\eta\bar{y}$ units.

The expected area of each triangle is then $\frac{\eta^2 \bar{y}}{2}$. The total number of unit years of NRFI held for the review period is $\frac{\eta^2 \bar{y}}{2}$ times the number of cycles of length η in one review period, $\frac{T}{\eta}$. Hence, the holding cost of NRFI per review period is

$$HC_T = h_2 \frac{\eta^2 \bar{y}}{2} \cdot \frac{T}{\eta} = \frac{h_2 \eta \bar{y} T}{2} .$$

Dividing by T yields the holding cost of NRFI per unit time:

$$HC = \frac{h_2 \eta \bar{y}}{2} . \qquad (1)$$

3.3.2.5 RFI_R Holding Cost

The dimensions of RFI_R holding cost, h_1, are dollars per unit year. Hence, to compute the RFI_R holding cost per unit time, the unit years of stock held must first be computed. Items are demanded at an average annual rate of $\bar{x}$ items. In the determination of NRFI holding cost, it was seen that the average number of items put into RFI_R stock per η was $\eta\bar{y}$. Because all items that are damaged or worn out cannot be repaired, we would normally expect the quantity demanded to exceed the quantity repaired. This information is portrayed in Figure 1.

The average amount of time, t_1, that $\eta\bar{y}$ will fill demands is $\frac{\eta\bar{y}}{\bar{x}}$. Thus, the holding cost per η is

$$HC_\eta = \frac{h_1 \eta \bar{y} t_1}{2} = \frac{h_1 \eta^2 \bar{y}^2}{2\bar{x}} ;$$

and the holding cost per unit time is

$$HC = \frac{h_1 \eta \bar{y}^2}{2\bar{x}} . \qquad (2)$$

3.3.2.6 RFI_P Holding Cost

Again, because of the dimensions of the holding cost, h_1, the unit years stocked must be computed. Rearranging the information in Figure 2 and including a buffer level, which is normally desirable when dealing with probabilistic demand, the RFI_P level may be portrayed as shown in Figure 4.

The inventory position, defined as the quantity of items on hand plus on order minus back orders, at the time a review is made is R. A time τ_P later, all units on order will have arrived and the inventory level will be R less the leadtime demand. The expected leadtime demand is $\tau_P(\bar{x} - \bar{y})$. Just prior to the arrival of the next procurement, one review period later, the inventory level will have decreased an amount equal to the period's demand. Hence, the inventory level just prior to the arrival of the next order is $R - \tau_P(\bar{x} - \bar{y}) - T(\bar{x} - \bar{y})$.

From the preceeding discussion of the RFI_R holding cost, it was noted that

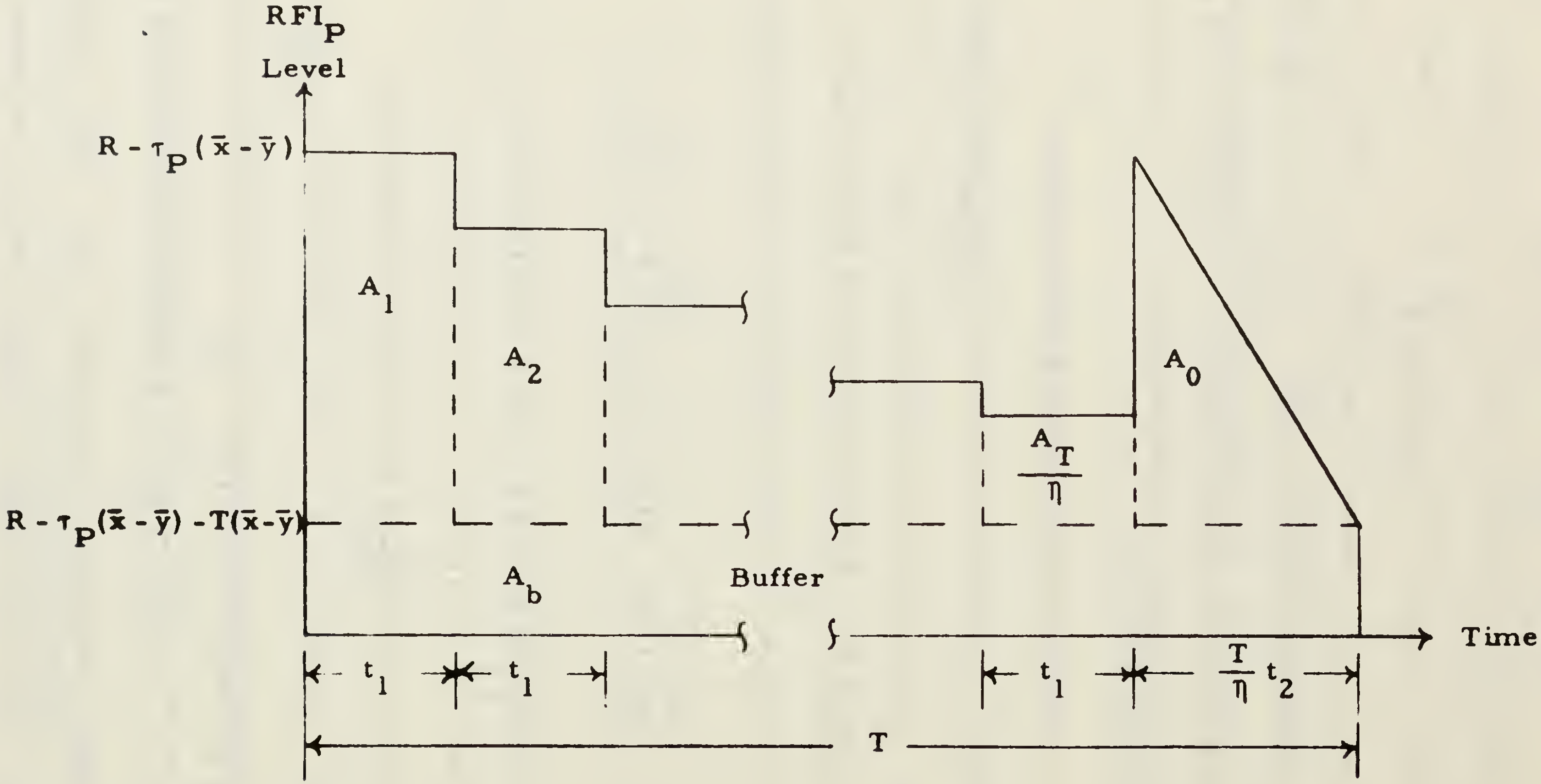
$$t_1 = \frac{\eta \bar{y}}{\bar{x}}$$

and

$$t_2 = \frac{\eta(\bar{x} - \bar{y})}{\bar{x}} .$$

Hence, the area of A_0 is

$$A_0 = \frac{T^2(\bar{x} - \bar{y})^2}{2\bar{x}} .$$



Rearrangement of Procured Ready-for-Issue Level Versus Time

FIGURE 4

During each period of η, the average number of items demanded upon RFI_P will equal the average number of demands that RFI_R could not satisfy, which is $\eta(\bar{x} - \bar{y})$. Therefore, the average height of each step in the step function in Figure 4 is $\eta(\bar{x} - \bar{y})$.

Each A_i, $i = 1, 2, \ldots, \frac{T}{\eta}$, will be the product of its height times t_1. Thus,

$$A_1 = [T(\bar{x} - \bar{y})] \frac{\eta \bar{y}}{\bar{x}} \quad ,$$

$$A_2 = [T(\bar{x} - \bar{y}) - \eta(\bar{x} - \bar{y})] \frac{\eta \bar{y}}{\bar{x}} \quad ,$$

$$A_3 = [T(\bar{x} - \bar{y}) - 2\eta(\bar{x} - \bar{y})] \frac{\eta \bar{y}}{\bar{x}} \quad ,$$

and, in general,

$$A_i = [T(\bar{x} - \bar{y}) - (i - 1)\eta(\bar{x} - \bar{y})] \frac{\eta \bar{y}}{\bar{x}} \quad .$$

The area of the buffer is

$$A_b = [R - \tau_P(\bar{x} - \bar{y}) - T(\bar{x} - \bar{y})] T \quad .$$

The total area, A_T, is determined by summing the various areas: the area of the triangle, A_0; the buffer area, A_b; and the areas of the rectangles, A_i, $i = 1, 2, \ldots, \frac{T}{\eta}$. Using the relations

$$\sum_{j=1}^{N} P = PN$$

and

$$\sum_{j=1}^{N} j = \frac{N(N + 1)}{2} \quad ,$$

the total area A_T may be determined to be

$$A_T = \frac{T(\bar{x} - \bar{y})(\eta\bar{y} - \bar{x}T)}{2\bar{x}} + T[R - \tau_P(\bar{x} - \bar{y})] .$$

One notes that the upper limit on the summation of the A_i's is $\frac{T}{\eta}$. This point may be questionable since there is no guarantee that $\frac{T}{\eta}$ will be an integer. However, if η is small compared to the review cycle, T, the error from considering $\frac{T}{\eta}$ as an integer is negligible.

The holding cost of RFI_P per review period is then $h_1 A_T$ and, dividing by the review period T, yields the holding cost per unit time.

$$HC = h_1 \left[R + \frac{(\bar{x} - \bar{y})(\eta\bar{y} - T\bar{x} - 2\bar{x}\tau_P)}{2\bar{x}} \right] . \qquad (3)$$

Additionally, there may be some question as to whether or not the holding cost should be modified to account for the unit years of items backordered. Because this is an approximate model, we ignore this term. In light of the objective of this paper, this assumption is only valid when the budget constraint is not too restrictive.

3.3.2.7 Units Backordered

A procurement order placed at time t will arrive at $t + \tau_P$, and the next order will arrive at time $t + \tau_P + T$. At time t, after the order is placed, the inventory position is R. The next time the inventory position reaches R is at time $t + \tau_P + T$. Hence, shortages will occur if the demand on RFI_P during $\tau_P + T$ exceeds R. The number of units demanded on RFI_P is $X - Y$. Let the random variable Z, with density function $h(z, t)$, equal $X - Y$. Now, the number of units backordered, V, is

$$V = \begin{cases} 0 & \text{if } Z \le R \\ Z - R & \text{if } Z > R \end{cases} .$$

Hence, the expected number of units backordered per unit time, denoted as S, is

$$S = \frac{1}{T} \int_R^{\infty} (Z - R)\, h(z, \tau_P + T)\, dz \quad . \tag{4}$$

This expression only accounts for the expected number of units backordered at the end of the time period $\tau_P + T$, and hence is only an approximation to the true expected number of units backordered during $\tau_P + T$. It does not consider the possibility that units may be backordered at the end of each time period η, and then filled by the input of repaired items into RFI_R. This possibility has been neglected since it is assumed that if this occurs, the number of items backordered and the length of time they are backordered are negligible. The duration of such shortages is only a fraction of η, which is assumed to be quite small. It should be noted that if the budget constraint is very restrictive, this assumption becomes unrealistic.

3.3.3 Solutions

As stated previously, the total variable operating cost per unit time of the system must be determined in order to minimize S, subject to a budget constraint. The variable cost per unit time, K, to operate the system is the sum of the various costs previously derived.

$$K = \frac{A_P + J}{T} + \frac{A_R}{\eta} + \frac{\eta \bar{y}(h_1 + h_2)}{2} + h_1 \left[R - \frac{(\bar{x} - \bar{y})(T\bar{x} + 2\bar{x}\tau_P)}{2\bar{x}} \right] . \quad (5)$$

Because demand is probabilistic and the most commonly used functions to describe demand do not have a finite upper bound, one would normally assume that the budget constraint would be active.

Assuming that the budget constraint is active, the normal method of solving the problem is through the use of the Lagrange multiplier. The general Lagragian equation is

$$L = S + \Pi(K - B) \quad .$$

The solution comes from meeting the following conditions:

$$\frac{\partial L}{\partial \Pi} = 0, \quad \frac{\partial L}{\partial R} = 0, \quad \frac{\partial L}{\partial T} = 0, \quad \frac{\partial L}{\partial \eta} = 0 \quad .$$

First, we will determine the optimal value of η by solving the $\frac{\partial L}{\partial \eta} = 0$ equation for η.

$$\frac{\partial L}{\partial \eta} = \frac{\partial S}{\partial \eta} + \Pi \frac{\partial K}{\partial \eta} = 0$$

and thus

$$\frac{\partial K}{\partial \eta} = - \frac{1}{\Pi} \frac{\partial S}{\partial \eta} .$$

From equation 4, we note that S is not a function of η ; hence, $\frac{\partial S}{\partial \eta} = 0$.

From equation 5, we see that $\frac{\partial K}{\partial \eta}$ is

$$\frac{\partial K}{\partial \eta} = - \frac{A_R}{\eta^2} + \frac{\bar{y}(h_1 + h_2)}{2} .$$

Solving the $\frac{\partial K}{\partial \eta} = 0$ for η yields

$$\eta = \left[\frac{2 A_R}{\bar{y}(h_1 + h_2)} \right]^{1/2} . \qquad (6)$$

Next, we shall look at the condition $\frac{\partial L}{\partial \Pi} = 0$. From the general Lagragian equation the $\frac{\partial L}{\partial \Pi} = K - B$. This implies that $K = B$. Hence, solving $K = B$ for R as a function of T yields

$$R = \frac{1}{h_1} \left[B - \frac{(A_P + J)}{T} - \frac{A_R}{\eta} - \frac{\eta \bar{y}(h_1 + h_2)}{2} \right] + \frac{(\bar{x} - \bar{y})(T\bar{x} + 2\bar{x}\tau_P)}{2\bar{x}} . \qquad (7)$$

To determine the optimal values of R and T one could solve the $\frac{\partial L}{\partial T} = 0$ and the $\frac{\partial L}{\partial R} = 0$ simultaneously with equation 7. However, it is quite easy to solve by selecting several values of T, computing the associated values of R from equation 7, and then determining the values of S from equation 4 using the values of R and T . Throughout these computations, the optimal value of η should be used. A plot of S versus T may then be made to indicate the value of T that minimizes S. With that value of T, equation 7 is then evaluated for R.

The interpretation of the Lagrange multiplier, Π, is that it represents the decrease in the expected number of units backordered per year for a unit increase in the budget. The value of Π may be obtained from the solution of the $\frac{\partial L}{\partial R} = 0$. This yields

$$\Pi = \frac{1}{h_1 T} \int_R^{\infty} h(z, \tau_P + T)\, dz \quad . \tag{8}$$

After obtaining the optimal values of R and T that yield the minimum expected number of units backordered, Π may be evaluated from equation 8.

3.3.4 Example

The following example is used to demonstrate the nature of the solutions presented by the model, and to explore the trade-off between procurement order and review costs and holding costs.

Let demand and the quantity of items returned to RFI_R be normally distributed with means of 1,000 units per year and 900 units per year, and standard deviations of 30 units per year and 50 units per year, respectively. Procurement lead time, τ_P, is .5 years. The relevant costs are: $A_P = \$750$, $A_R = \$100$, $J = \$250$, $h_1 = \$200$, and $h_2 = \$20$. The two random variables, X and Y, are assumed to be independent. Also, demand during T is independent of demand during τ_P; and the quantity of items returned to RFI_R during T is independent of the quantity of items returned to RFI_R during τ_P. Therefore, the random variable Z is normally distributed with mean $100(\tau_P + T)$ and standard deviation equal to $[(\tau_P + T)\,3400]^{1/2}$.

Let B = \$20,000. Following the procedure outlined in section 3, the first step is to determine η. Evaluating equation 6 yields $\eta = 3.18 \times 10^{-2}$ years, which is approximately 12 days. Next, we solve equation 7 for various values of R for selected values of T. If T = .1 years, R = 73 units; if T = .5 years, R = 133 units; and if T = 1 year, R = 163 units. Next, using these values of R and T, determine the associated values of S. For this example, the general solution of S is

$$S = \frac{1}{T}\left[\sigma\,\phi\left(\frac{R-\bar{z}}{\sigma}\right) + (\bar{z} - R)\,\Phi\left(\frac{R-\bar{z}}{\sigma}\right)\right] ,$$

where

$$\phi(r) = \frac{1}{\sqrt{2\Pi}}\, e^{-\frac{r^2}{2}} ,$$

$$\Phi(r) = \int_r^{\infty} \phi(x)\, dx ,$$

and $\bar{z}$ is the mean of Z, and σ is the standard deviation of Z.

The associated values of S are as follows:

T	R	S
0.1	73	123
0.5	133	21
1.0	163	22

From plotting these values, we see that the minimum S occurs near T = .8 years. Evaluating R for T = .7, .8, and .9 years and evaluating the associated values of S, the following results were obtained.

T	R	S
0.7	146	21
0.8	152	20
0.9	158	22

Thus, the minimum expected number of units backordered per year is 20 when the review cycle is .8 years, the procurement order up to level is

152 units, and the cycle time for the receipt of repaired items is 3.18×10^{-2} years. The Lagrange multiplier, Π, may now be evaluated from equation 8. This yields $\Pi = 2.31 \times 10^{-3}$ units per dollar.

Various budget levels were considered to investigate the changes in the expected number of units backordered per year, S. The results are tabulated below. The mean demand during $\tau_P + T$, denoted as $\bar{\bar{x}}$, and the expected number of units backordered per review period, denoted as S_T, are also shown.

B	T	R	$\bar{\bar{x}}$	S	S_T
$10,000	4.0	276	450	45	180
$15,000	0.9	133	140	35	32
$20,000	0.8	152	130	20	16

The cost to operate the repair side of the system per year, with no review and procurement order costs, is approximately $6,000. With this budget level, the expected number of units backordered per year is 100. When the budget is increased to $10,000, the review cycle is very long because it is profitable to put the majority of the additional money into holding procured items, instead of sustaining the high procurement order and review costs frequently. As the budget is increased to $15,000, the review cycle decreases to .9 years, indicating that the system can afford to review and order procurement more often in order to achieve an economic balance between ordering and reviewing and holding costs. An increase of the budget by one-third to $20,000 yields only a slight decrease of the review period to .8 years, thereby putting most of the increased budget into safety stock.

3.3.5 Summary

We have discussed a reparable item inventory system with random demand. A periodic review policy was assumed for procured items, while inductions of carcasses were assumed to take place at regularly spaced intervals. As the accumulation rate of NRFI carcasses was assumed random, the repair batch sizes were also random. The objective was to determine the optimal procurement review period T, the optimal procurement order up to level R, and the optimal repair time period η, in order to minimize the expected number of units backordered per unit time, subject to an annual operating budget constraint. The model developed is an approximate model. The solutions presented are sensitive to the assumptions that η is small compared to T, and that the expected unit years of items backordered are small. Obviously, the model is also sensitive to the restrictiveness of the budget constraint.

As discussed in the introduction, the purpose of minimizing S subject to a budget constraint was to avoid postulating a shortage cost. However, even though a shortage cost may not be stated by a decision-maker or an item manager, it is implied as soon as a specific operating budget is established. We noted in section 3 that Π, defined as the

Lagrange multiplier, may be interpreted as the decrease in the expected number of units backordered per year for a unit increase of the budget. Therefore, $\frac{1}{\Pi}$ is the shortage cost, which would yield the same decision rules if the more common criteria of minimum cost per unit time were used.

3.4 A Periodic Reparable-Item Inventory Model

3.4.1 Introduction and Notation

An existing reparable inventory system within the Naval Aviation Supply System is an exceedingly complex probabilistic system involving tens of thousands of line items and a network of stock points and repair points. Anyone attempting to model a system of this complexity is immediately aware that the model can never be more than a partial representation of reality. Despite this awareness of incompleteness, the fact that models have proven to be invaluable aids in managing complex systems stands as a primary motivating force behind the efforts to model a reparable inventory system.

The purpose of this paper is to add to this growing knowledge base by structuring a periodic review model for a reparable item system experiencing normally distributed random demands. The model, in turn, provides a means of understanding the parameter interactions and cost-protection trade-offs that exist within a system of this nature. The periodic model techniques used by Hanssman [22] in the development of a multilevel production control model are relied upon during the initial phase of the model development.

In addition to the model's basic assumptions of periodic review and normal demand, it was necessary to further simplify the reparable item system by making additional assumptions. The major assumptions are included in the following summary of the scope of the model.

The model addresses the reparable item system on a single item basis. The system is considered to be made up of two subsystems: (1) the repaired item subsystem, and (2) the purchased (new) item subsystem. The purchased item subsystem is made up of a single manufacturer, the outstanding purchase order quantities, and the purchased item portion of the ready-for-issue (RFI) inventory under the control of the inventory control point (ICP). The repaired item subsystem is made up of a single repair facility (with an inventory of non-ready-for-issue (NRFI) items), outstanding repair order quantities, and the repaired item portion of the RFI inventory under the control of the same ICP. The total demand each period is apportioned into two segments based on the recovery rate r; i.e., r percent of the demand is charged against the ICP repaired item inventory and the remaining portion is satisfied from the ICP purchased item inventory. The assumption is made that the recoverable portion of the items demanded each period will be returned to the repair facility in a given number of periods after the period in which they were demanded.

The basic model assumes that repair and purchase orders are initiated each period by the ICP. A sufficient quantity is ordered each period to bring the subsystems' inventory positions up to established limits. Separate net inventory distributions are developed for the ICP repaired item inventory, ICP purchase item inventory, ICP combined purchased and repaired item inventory, and the repair facility inventory. The relationship between the high limits of the subsystem and the corresponding protection level, i. e., one minus the probability of stockout, are determined and corresponding system-cost equations are derived.

The equations of the basic model are then slightly modified to allow for extension of the repair and purchase review periods in multiples of the basic review period. The extension does not require that these two review periods be of equal length, as was assumed in the basic model. The annual operating cost trade-off that exists between (1) period review and order costs and (2) ICP inventory holding costs for a given protection level is examined. A method for determining the operating procedures that minimize the expected annual operating cost, within the evaluation restrictions of the extended model, are then investigated.

The following notation will be used throughout this section.

R - Repair order lead time (periods).

L - Purchase order lead time (periods).

X_i - Random variable representing system demand during period i.

σ_X^2 - Variance of X.

μ_X - Mean of X.

$I_2(i)$ - Random variable representing repaired item inventory level at inventory control point (ICP) at the end of period i.

σ_2^2 - Variance of I_2.

μ_2 - Mean of I_2.

$I_1(i)$ - Random variable representing ICP purchased item inventory level at the end of period i.

σ_1^2 - Variance of I_1.

μ_1 - Mean of I_1.

V - Random variable representing non-ready-for-issue (NRFI) inventory level at the repair activity.

σ_V^2 - Variance of V.

μ_V - Mean of V.

$I(i)$ - Random variable representing total inventory level $(I_1 + I_2)$ under the control of ICP at the end of period i.

σ^2 - Variance of I.

μ - Mean of I.

r - Fixed recovery rate (percentage).

α - Protection level.

Q_i - Quantity requested by ICP repair order submitted at the beginning of period i.

q_i - Batch quantity that the repair facility started repairing at the beginning of period i.

P_i - Quantity requested by ICP purchase order submitted at the beginning of period i.

K^1 - Average number of periods to release full order at repair facility.

K - Turn around time.

A_P - Fixed procurement order cost (per order).

A_R - Fixed repair order cost (per order).

h - Ready-for-issue (RFI) inventory holding cost per item per period at ICP.

h_V - NRFI inventory holding cost per item per period at repair activity.

H_1 - ICP high limit for the purchased-item system.

H_2 - ICP high limit for the repaired-item system.

T - Basic model's period length (in year units).

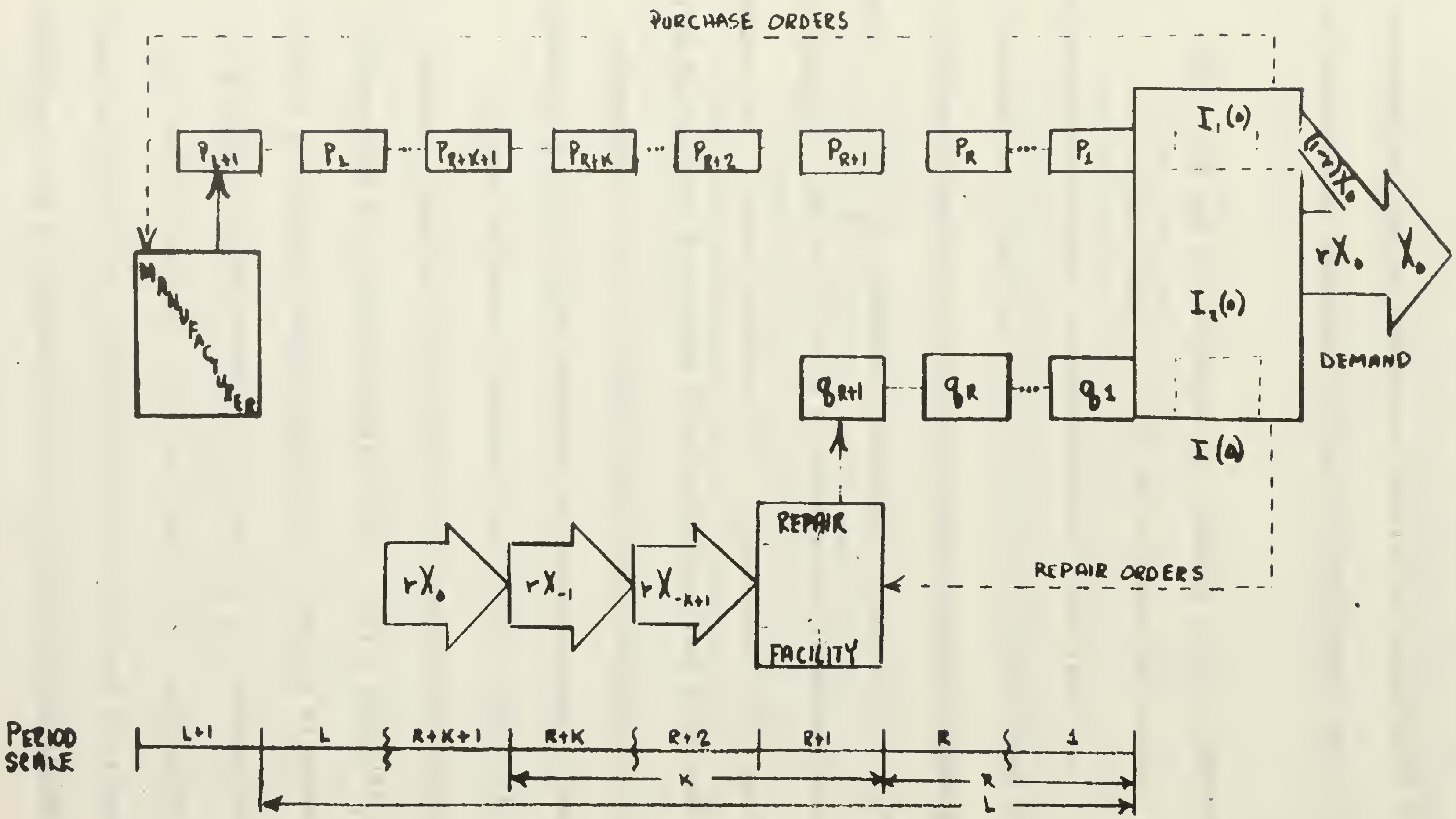
3.4.2 A Periodic Reparable Model

3.4.2.1 Description of Reparable Item System

Figure 1 illustrates the periodic structure and relative leadtime relationships of the reparable item system addressed and modeled by this paper.

The description of the system will commence at the ICP. The inventory, I, under the control of the ICP is looked upon as two inventories; i.e., those items received directly from the manufacturer, I_1, and those items received from the repair facility, I_2. The total withdrawal quantity (demand) during period i is a random variable denoted by X_i. It is assumed that period demands, $X_1, X_2, \ldots$, are independent and identically distributed as a normal random variable X having mean μ_X and variance σ_X^2. The recovery rate, r, is a known percentage; that is, r percent of the items demanded each period will be returned in a reparable state to the repair facility. Each period rX_i of the total items demanded are satisfied from I_2 and the remaining portion, $(1 - r)X_i$, of the total demand is satisfied from I_1.

At the end of period zero, the combined ICP inventory position, I(0), consists of $I_1(0)$ plus $I_2(0)$. The outstanding orders at this moment in time are shown in Figure 1. The repair order quantity q_i and the purchase order quantity P_i go into ICP inventory at the beginning of period i. The purchase order lead time is L periods and the repair order lead time is R periods, where lead time is defined as the fixed interval of periods between the time the ICP decides to place an order



Flow Diagram of Basic Periodic Review Model

FIGURE 1

and the time the ordered quantity is placed in the ICP controlled inventory. Just prior to the beginning of period 1, the new orders, Q_{R+1} and P_{L+1}, must be determined.

It is assumed that at the beginning of each period the repair facility receives the reparable portion of the system demand experienced by the system K periods earlier, where K is defined as the time in periods from when an item is demanded and the replaced item is returned to the repair facility. An ICP repair order Q is received by the repair facility at the beginning of each period. Upon receipt of the order, the repair facility immediately commences a batch repair to fill the order with the existing stock on hand. If there is insufficient stock on hand to meet the order at this point in time, the unfilled portion of the new order is included in the batch repair initiated at the beginning of the next period.

3.4.2.2 ICP Purchased Inventory Distribution and Order Rules

At the end of each period, a new purchase order, P_{L+1}, is determined by the ICP and transmitted to the manufacturer. It is assumed that the manufacturer has sufficient raw material on hand to commence production of the ordered items immediately upon receipt of the ICP order. Therefore, the first I_1 inventory level that the ICP can control is $I_1(L+1)$, where

$$I_1(L+1) = I_1(0) + P_1 + P_2 + \dots + P_L + P_{L+1} - ((1-r)X_1 + \dots + (1-r)X_{L+1}) \quad . \quad (1)$$

Let H_1 be defined as the high limit of the purchased item system, which is a fixed value to be determined by management. This limit represents the maximum level of items in the purchased item system, i.e., ICP purchased item inventory and outstanding purchase order quantities. Each period an order is placed to bring the purchased item system back up to this level. It follows that

$$H_1 = I_1(0) + \sum_{i=1}^{L} P_i + P_{L+1} \tag{2}$$

or

$$P_{L+1} = \text{new order} = H_1 - I_1(0) - \sum_{i=1}^{L} P_i \ .$$

Based on equation (2), we can write (1) as follows:

$$I_1(L+1) = H_1 - \sum_{i=1}^{L+1} (1-r) X_i \ . \tag{3}$$

As shown in Appendix A, I_1 is a normally distributed random variable with parameters

$$\mu_1 = H_1 - (L+1)(1-r)\mu_X \tag{4}$$

and

$$\sigma_1^2 = (L+1)(1-r)^2 \sigma_X^2 \ , \tag{5}$$

where a negative inventory is considered to be an outstanding backorder quantity when the on-hand inventory level is zero.

The value of the mean parameter μ_1 can also be written as a function of a desired inventory protection level. Given a desired protection level α, the corresponding probability of stockout is $1 - \alpha$, which can be written

$$P[I_1 \leq 0] = 1 - \alpha \quad .$$

This, in turn, can be written in the standard normal distribution function form, $\Phi(i)$, as follows:

$$\Phi\left(\frac{0 - \mu_1}{\sigma_1}\right) = 1 - \alpha$$

or

$$\Phi\left(\frac{\mu_1}{\sigma_1}\right) = \alpha \quad .$$

Defining $N(\alpha)$ as the value obtained from a standard normal distribution table corresponding to the area α under the curve,

$$N(\alpha) = \frac{\mu_1}{\sigma_1} \quad . \tag{6}$$

Rearranging and replacing σ_1 with its equivalent form shown in (5), this can be written

$$\mu_1 = N(\alpha)\,(L + 1)^{1/2}\,(1 - r)\,\sigma_X \quad . \tag{7}$$

From equations (7) and (4), the value of the high limit of the purchase system H_1 can be expressed as a function of the desired protection level α and the parameters μ_X and σ_X.

$$H_1 = (1 - r)\,\left(N(\alpha)\,(L + 1)^{1/2}\,\sigma_X + (L + 1)\mu_X\right) \quad . \tag{8}$$

3.4.2.3 ICP Repaired Inventory Distribution and Order Rules

The repair facility may not be able to start on a full ICP order due to an insufficient NRFI inventory of reparable carcasses on hand. The point in time when a new repair order, Q_{R+1}, will affect the inventory at the ordering activity depends upon the lead time and the amount of shortage at the repair facility receiving the order. As pointed out by Hanssman [5], this shortage is characterized by the average time K^1 it takes the repair facility to release the full amount of the order quantity Q_{R+1}. Referring to Figure 1, the actual output by the repair facility into q_{R+1} will be equal to the total unfilled orders or the total supply on hand, whichever is smaller. It is assumed that K^1 is an integer multiple of the unit period. Therefore, the first I_2 inventory level that the ICP can control is

$$I_2(R + K^1 + 1) \quad .$$

Now,

$$I_2(R + K^1 + 1) = I_2(0) + Q_1 + Q_2 + \cdots + Q_R + Q_{R+1} - (rX_1 + \cdots + rX_{R+K^1+1}) \; . \quad (9)$$

Let H_2 be defined as the high limit of the repaired item system which is a fixed value to be determined by management. This limit is the maximum level of items represented by the ICP repaired item inventory and outstanding repair order quantities. (The repair facility inventory is not considered within this limit.) Each period an order is placed to bring the repaired item system back up to this level. It follows that

$$H_2 = I_2(0) + \sum_{i=1}^{R} Q_i + Q_{R+1}$$

or (10)

$$Q_{R+1} = \text{new order} = H_2 - I_2(0) - \sum_{i=1}^{R} Q_i \quad .$$

Based on (10), we can write (9) as follows:

$$I_2(R + K^1 + 1) = H_2 - \sum_{i=1}^{R+K^1+1} r X_i \quad . \tag{11}$$

From (9), it can be shown (similar to the Appendix A derivation) that I_2 is a normally distributed random variable with parameters

$$\mu_2 = H_2 - (R + K^1 + 1) r \mu_X \tag{12}$$

and

$$\sigma_2^2 = (R + K^1 + 1) r^2 \sigma_X^2 \quad , \tag{13}$$

where a negative inventory is considered to be the outstanding backorder quantity when the on-hand inventory level is zero.

By the method shown in the equation (7) derivation, the value of the mean parameter μ_2 can be written as a function of a desired protection level α as follows:

$$\mu_2 = N(\alpha)\,(R + K^1 + 1)^{1/2}\, r\,\sigma_X \quad . \tag{14}$$

From equations (12) and (14), the value of the high limit of the repair system H_2 can be expressed as a function of the desired protection level α and the parameters μ_X and σ_X .

$$H_2 = r\left[N(\alpha)\,(R + K^1 + 1)^{1/2}\,\sigma_X + (R + K^1 + 1)\mu_X \right] . \qquad (15)$$

3.4.2.4 ICP Combined Inventory Distribution

Having determined the distributions of I_1 and I_2 , the parameters of the combined inventory distribution, $I = I_1 + I_2$, can now be determined. Due to the common demand distribution from which I_1 and I_2 were derived, a dependent relationship exists between I_1 and I_2 . Under these conditions, the parameters of the combined inventory distribution, I, are defined [23] as follows:

$$\mu = \mu_1 + \mu_2 \qquad (16)$$

and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\,\mathrm{COV}\,(I_1 , I_2) \quad . \qquad (17)$$

Under the assumption that the joint distribution is normal, it follows that the combined inventory distribution, I, is also normally distributed with the parameters as shown above. An alternative form of the mean parameter can be obtained from (7) and (14) and the assumption that the same protection level α will be set for both the I_1 and I_2 inventories.

$$\mu = N(\alpha)\,\sigma_X \left((L + 1)^{1/2} (1 - r) + (R + K^1 + 1)^{1/2} r \right) . \qquad (18)$$

From (5), (13), and the derivation of COV (I_1 , I_2) in Appendix B, the variance parameter can be expressed as follows:

$$\sigma^2 = ((L+1)(1-r)^2 + (R+K^1+1)r^2 + 2r(1-r)(R+K^1+1))\sigma_X^2 \quad . \qquad (19)$$

The above variance equation only holds if it is assumed that leadtime relationships satisfy

$$L \geq R + K^1 \quad .$$

Further, although this equation gives insight as to the effect of the various system parameters upon the combined inventory variance, it cannot be used to develop a single protection level relationship unless the separate I_1 and I_2 inventory assumption is relaxed. To relate a protection level to this term (vice separately for I_1 and I_2) would imply that material from one inventory could satisfy demand being experienced by the other inventory. In terms of practical application of the model, this interchange would be a highly desirable extension; it will be discussed below.

3.4.2.5 Repair Facility Inventory Distribution

Similar to the I_1 demand derivation in Appendix A, it can be shown that the random demand, rX, charged against the I_2 inventory in a given period is normally distributed with parameters $(r\mu_X, r^2\sigma_X^2)$. Based on the repair order rules developed in subsection 2.2, the corresponding ICP repair order determined at the end of each period is equivalent to the repaired item demand during that period. It follows that the ICP repair orders can be thought of as being generated from the I_2 demand distribution given above. Further, the input into the repair facility each period (number of reparable carcasses returned) was assumed to be generated from the same demand distribution K periods earlier. Assuming that K is greater than one period, the repair facility's net inventory position each period is a random variable determined by the difference between these two independent (K periods apart) normal random variables. Given these conditions, it follows that the repair facility's inventory V is normally distributed with parameters defined as follows:

$$\mu_V = 0 \tag{20}$$

and

$$\sigma_V^2 = 2r^2\sigma_X^2 \quad , \tag{21}$$

where the net value of the inventory takes on negative values (backlog) when the ICP repair order exceeds the on-hand physical inventory.

In subsequent sections where inventory carrying costs are considered, it becomes necessary to determine the expected overage so that the average holding cost of the on-hand physical inventory can be estimated. This value is found in the following manner.

Defining the overage as

$$\text{overage} = \begin{cases} V & \text{if } V \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad ,$$

the expected overage is

$$E[\text{overage}] = \frac{1}{\sqrt{2\pi}\,\sigma_V} \int_0^{\infty} V \exp\left(-\frac{V^2}{2\sigma_V^2}\right) dV$$

$$= \frac{\sigma_V}{\sqrt{2\pi}} = \frac{r\sigma_X}{\sqrt{\pi}} \quad . \tag{22}$$

The expected shortage value can be determined in a similar manner. The average number of periods, K^1, it takes the repair facility to release the full amount of a repair order can be determined from the expected shortage value.

Defining the shortage as

$$\text{shortage} = \begin{cases} -V & \text{if } V \le 0 \\ 0 & \text{otherwise} \end{cases} ,$$

the expected shortage is

$$E[\text{shortage}] = -\frac{1}{\sqrt{2\pi}\,\sigma_V} \int_{-\infty}^{0} V \exp\left(-\frac{V^2}{2\sigma_V^2}\right) dV$$

$$= \frac{\sigma_V}{\sqrt{2\pi}} = \frac{r\sigma_X}{\sqrt{\pi}} \quad . \tag{23}$$

Since $r\mu_X$ is the mean input into the repair facility each period, the expected number of periods required to accumulate sufficient material to fill the expected shortage can be represented approximately by

$$\frac{E[\text{shortage}]}{r\,\mu_X} = \frac{\sigma_X}{\sqrt{\pi}\,\mu_X} \quad .$$

Obviously, the preceding quotient will include fractions of periods. However, in keeping with the subsection 3.4.2.3 derivation, it becomes necessary to treat a fraction of a period as an additional whole period. This represents a conservative approach in opposition to the alternative of dropping the fraction, i.e., K^1 will be a fraction of a period larger than the expected delay time vice a fraction smaller. Accordingly, K^1 will be defined as follows:

$$K^1 = \left[\frac{\sigma_X}{\sqrt{\pi}\ \mu_X} \right] + 1 , \qquad (24)$$

where [ ] is defined as the greatest integer in the enclosed value.

3.4.2.6 Operating Cost Considerations

As pointed out in [1], there are numerous costs associated with the management of a reparable inventory system. Since the construction of a mathematical model of an inventory system is motivated by a desire to improve the operating rules for controlling a particular inventory system, the pertinent costs are those which are influenced by the operating doctrine. It follows that costs that are independent of the operating doctrine need not be included in the analysis. The periodic review costs considered herein have been so restricted and, wherever possible, have been grouped under one symbol to reduce the model's notation complexity. The costs are defined as follows:

(a) Purchase Order Cost (A_P). The fixed costs associated with (1) reviewing the purchase order records and (2) preparing a purchase order.

(b) Repair Order Cost (A_R). The fixed costs associated with (1) reviewing the repair order records, (2) preparing a repair order, and (3) initiating batch repair action at the repair facility in response to a repair order (regardless of size).

(c) ICP Holding Cost (h). The cost per unit per basic review period to hold a RFI item in the ICP inventory. This cost is normally a function of item purchase cost and an estimated holding rate.

(d) Repair Holding Cost (h_V). The cost per unit per basic review period to hold a NRFI item at the repair facility. This cost is normally a function of the NRFI item value and an estimated holding rate.

Since the net inventory distributions are subsequently used to determine the value of the holding costs, it is emphasized that these distributions were derived by treating the random demand each period in a discrete manner, i.e., as if the total period demand was received at a point in time within the period. Consequently, a holding cost is incurred only if the period's on-hand inventory plus receipts exceed period demand. The period holding cost is then proportional to the excess stock on hand.

A specific shortage cost has not been included in the analysis; however, a shortage cost is implied by the decision-maker's choice of protection level. In this regard, it is envisioned that management would be presented with yearly cost estimates for various protection levels and that the final protection level decision would be influenced by the associated costs. Accordingly, the primary objective of deriving a cost expression for the basic model is to provide a means of relating different protection levels to the expected annual operating costs corresponding to these levels of protection.

The analysis will be based on the periodic structure of the basic model presented in subsection 3.4.2.1, where the purchase and repair orders are initiated at the end of each period. Defining T as the basic period length in year units, the average annual purchase order cost is $\frac{A_P}{T}$. Similarly, the average annual repair order cost is $\frac{A_R}{T}$.

The expected period cost of holding inventory at the ICP is found by multiplying the ICP holding cost (h) times the expected value of the net inventory at the ICP. Assuming that management will set the protection level for the I_1 and I_2 net inventories sufficiently high so that back orders are incurred only in small quantities, the expected values of the net inventories will very closely approximate the expected values of the on-hand physical inventories. Accordingly, the mean value μ (where $\mu = \mu_1 + \mu_2$) of the combined ICP inventory, I, will be used to approximate the expected value of the physical inventory. The expected holding cost per period is, therefore, $h\mu$ and the expected yearly cost is $\frac{h\mu}{T}$.

The expected period cost of holding inventory at the repair facility is found by multiplying the repair holding cost (h_V) times the expected value of the physical inventory at the repair facility. Here it cannot be assumed that back orders are incurred in small quantities. As a result of the derivation in subsection 3.4.2.5, a back order or overage position is equally likely at the repair facility. Using the expected value of the physical inventory (expected overage) developed in subsection 3.4.2.5, the expected period cost of holding inventory at the repair facility is

$$\frac{h_V \, r \, \sigma_X}{\sqrt{\pi}} \; .$$

The expected annual cost of holding inventory at the repair facility is then

$$\frac{h_V \, r \, \sigma_X}{T \sqrt{\pi}} \; .$$

All the terms to be considered in the expected annual cost expression of the basic model have now been evaluated.

$$\text{Annual Cost} = \frac{1}{T} \left(A_P + A_R + h\mu + \frac{h_V \, r \, \sigma_X}{\sqrt{\pi}} \right) . \tag{26}$$

Or, replacing μ with its equivalent (18) form, the annual cost can be expressed as follows:

$$\text{Annual Cost} = \frac{1}{T} \Big(A_P + A_R + h N(\alpha) \sigma_X \big((L+1)^{\frac{1}{2}} (1 - r) + (R + K^1 + 1)^{\frac{1}{2}} r \big) + \frac{h_V \, r \, \sigma_X}{\sqrt{\pi}} \Big) . \tag{27}$$

Equation (27) relates the annual operating costs with the protection level α. As α is increased, the value of $N(\alpha)$ will increase, causing the annual operating cost to increase. Since the basic model assumed a given review period length, the operating cost can only be varied by varying the protection level. Therefore, once the basic review period has been established, the decision-maker's final protection level choice will be governed by the associated annual operating costs he is willing to accept or, in the case of a budget constraint, the annual operating funds available.

3.4.3 Extension of the Basic Model

The structure of the basic model was based upon a given review period. Review and order action took place at the end of this unit period and ordered quantities were received at the beginning of the period. The following extension of the model to evaluate the effect of extending the ICP repaired-item review period and/or the purchased-item inventory review period will maintain the same relative time point relationships between these events within the extended periods. This requires that only those time periods that are integer multiples of the basic period and are evenly divisible into the original number of lead-time periods can be considered in the analysis.

Let T_1 be defined as an extended time period with length equal to an integer multiple of the basic time period such that $L \geq T_1 \geq 1$ and $\frac{L}{T_1}$ is an integer. Let T_2 be similarly defined as an integer multiple of the basic time period such that $R \geq T_2 \geq 1$ and $\frac{R}{T_2}$ is an integer. The demand over these extended periods is the sum of the independent basic period demands contained within the extended interval. It follows that the demand over the extended periods remains normally distributed. The mean and variance parameters of the extended period demand distributions can be obtained by scaling the corresponding basic model's I_1 and I_2 demand parameters (mean and variance) by the integer values of T_1 or T_2. The length of the purchase and repair order lead times in terms of the extended periods are represented by $\frac{L}{T_1}$ and $\frac{R}{T_2}$, respectively.

Using the preceding definitions, all the basic model equations in section 2 can be rederived in the same manner as before but in terms of the extended periods T_1 and T_2. A similar rederivation of the repair facility inventory distribution implies another restriction on the values that can be taken on by T_2. Recalling that the subsection 3.4.2.5 derivation assumed that the repair facility input was independent of the ICP repaired item demand during the previous period, it becomes necessary to further restrict T_2 to values less than the turn-around time K. Keeping in mind the restrictions on T_1 and T_2, the principle equations developed in section 2 can be rewritten as functions of T_1 and T_2 as follows.

(a) ICP Purchased Item Inventory (I_1)

$$\sigma_1^2 = (L + T_1)(1 - r)^2 \sigma_X^2 \tag{28}$$

$$\mu_1 = N(\alpha)(L + T_1)^{\frac{1}{2}} (1 - r)\sigma_X \tag{29}$$

$$H_1 = (1 - r)\left(N(\alpha)\sigma_X (L + T_1)^{\frac{1}{2}} + (L + T_1)\mu_X\right) \tag{30}$$

(b) ICP Repaired Item Inventory (I_2)

$$\sigma_2^2 = (R + K^1 T_2 + T_2) r^2 \sigma_X^2 \tag{31}$$

$$\mu_2 = N(\alpha)(R + K^1 T_2 + T_2)^{\frac{1}{2}} r \sigma_X \tag{32}$$

$$H_2 = r\,(N(\alpha)\,\sigma_X\,(R + K^1 T_2 + T_2)^{\frac{1}{2}} + (R + K^1 T_2 + T_2)\,\mu_X) \tag{33}$$

(c) Repair Facility Inventory (V)

$$\mu_V = 0 \tag{34}$$

$$\sigma_V^2 = 2\,T_2\,r^2\,\sigma_X^2 \tag{35}$$

$$E\,[\text{overage}] = \frac{(T_2)^{\frac{1}{2}}\,r\,\sigma_X}{\sqrt{\pi}} \tag{36}$$

$$K^1 = \left[\frac{\sigma_X}{(\pi\ T_2)^{1/2}\ \mu_X}\right] + 1 \tag{37}$$

(d) Expected Annual Operating Cost

$$\text{Annual Cost} = \frac{1}{T}\left(\frac{A_P}{T_1} + \frac{A_R}{T_2} + h\,N(\alpha)\,\sigma_X \cdot \left((L + T_1)^{\frac{1}{2}}\,(1 - r) + (R + K^1 T_2 + T_2)^{\frac{1}{2}}\,r\right) + \frac{h_V\,(T_2)^{\frac{1}{2}}\,r\,\sigma_X}{\sqrt{\pi}}\right) \tag{38}$$

The values of T_1 and T_2 that minimize the annual operating cost expression (38) for a given protection level (α) can be determined in the following manner:

(1) Hold T_2 constant at a value of one. Solve (38) for all integer values of T_1 permitted by the model. Select that value of T_1 which minimizes (38).

(2) Hold T_1 at the value determined above. Solve (38) for all integer values of T_2 permitted by the model. Select that value of T_2 which minimizes (38).

Using the above values of T_1 and T_2, the corresponding high limits H_1 and H_2 can be calculated from (30) and (33).

3.4.4 Example

The following numerical example is used to indicate the nature of the solutions given by the model. The parameter values chosen are as follows:

$$T = 1/12 \text{ year}$$

$$\mu_X = 40$$

$$\sigma_X = 4$$

$$r = .9$$

$$L = 9 \text{ months}$$

$$R = 4 \text{ months}$$

$$K = 5 \text{ months}$$

$$A_P = \$200$$

$$A_R = \$100$$

$$h = \$20 \text{ per item - month}$$

$$h_V = \$10 \text{ per item - month}$$

$$\alpha = .95 \text{ or } .99$$

With these values, the following results were obtained:

$\underline{95\% \text{ Protection } (\alpha = .95)}$	$\underline{99\% \text{ Protection } (\alpha = .99)}$
$N(\alpha) = 1.65$	$N(\alpha) = 2.326$
$K^1 = 1$	$K^1 = 1$
$T_1 = 9$ (months)	$T_1 = 9$ (months)
$T_2 = 1$ (month)	$T_2 = 1$ (month)
$H_1 = 74.8$	$H_1 = 75.9$
$H_2 = 229.6$	$H_2 = 236.5$
Annual Cost = \$5,873	Annual Cost = \$7,580

The costs per year of operating the inventory system under the basic model ($T_1 = 1$, $T_2 = 1$) were calculated to be \$7,835 and \$9,471 for the 95% and 99% protection levels, respectively. The combination of $T_1 = 9$ and $T_2 = 1$ produced the minimum annual costs under both the selected protection levels as shown above. The other combinations permitted by the model, i.e., $T_1 = 1, 3, 9$, $T_2 = 1, 2, 4$, generated higher annual costs.

The model is highly sensitive to the standard deviation of demand, σ_X. For example, if σ_X was actually 8 vice 4 units as used in the above example, the 99% protection level would be reduced to 87% (holding the cost and other operating variables constant). Retaining the same review period combination $T_1 = 9$ and $T_2 = 1$, this sensitivity can be illustrated in another manner. If σ_X were raised from 4 to 8 units, the same 99% protection level computation would cause the expected annual operating cost to almost double from \$7,580 to \$14,891.

The sensitivity of the model to the repair and purchase order lead times, R and L, depends upon the values of σ_X and r. As σ_X increases, the model becomes more sensitive to the lead times which are scaled by r and (1 - r), respectively. For example, if the repair order lead time for the 99% protection example was reduced from 4 to 2 months, the expected annual operating cost would be reduced only \$903 from \$7,580 to \$6,677. However, if σ_X were 8 vice 4 units, this same reduction in lead time would cause the expected annual operating cost to be reduced \$3,004 from \$14,891 to \$11,887.

The selection of the protection levels (95% and 99%) for the examples was arbitrary. A family of solutions could be obtained for various protection levels and presented to management for the selection of that protection level that best met the funding constraint for the particular item in question.

The fact that the example's minimum cost was found at $T_1 = 9$ points up the limitation of the extended model. This value represents an end-point solution in that this is the maximum value that can be taken on by T_1 within the extended model structure. The possibility that additional savings might be realized by extending the review period length even further remains unresolved. Looking in the other direction, it is possible to evaluate the effects of period lengths smaller than the example's basic unit period T by equating T to a smaller period, i. e., a week vice a month, and using correspondingly smaller demand data.

Nevertheless, the relatively few discrete points at which annual operating costs can be evaluated remains a limiting characteristic of the model.

3.4.5 Conclusions and Recommendations

As a result of the assumptions and the evaluation restrictions of section 3.4.3, the model's scope of application is necessarily confined to particular cases. Within this scope, the equations provide a means of obtaining an understanding of the iteractions of the principle parameters and how they affect the inventory positions and costs associated with operating a system of this nature. The model's equations indicate the sensitivity of the reparable-item system inventories to order lead times, return rates, mean demand, and protection levels. More significantly, the equations highlight the high sensitivity of the system to the demand variance parameter.

Possible courses of action to improve the model of the periodic review system addressed by this paper include the following:

(1) Define an acceptable method and/or criterion whereby material from either one of the two ICP inventories treated by this paper could be used to satisfy demand experienced by the other. This could be permitted in the present basic model if one accepts the rationale that the ordering rules are based upon "paper" inventory positions and that the physical inventories could be co-mingled. This would permit the use of the basic model's combined inventory variance term (19) in an overall

protection level computation. However, under the present structure, the covariance between the two inventories would be difficult to evaluate in the extended model cases due to the changing dependence relationships between repaired and purchased item demand over varying period lengths.

(2) Modify the periodic review structure in such a manner that the integer restrictions that confine the present model could be relaxed. This would permit finer resolution of the least cost review period combination and more exact treatment of the expected delay time K^1 at the repair facility.

This paper looked upon the reparable item system as two subsystems. "Optimizing" this type of structure eventually leads to combining sub-optimized parts vice truly optimizing the whole system. However, if one subsystem is significantly larger than the other, the shortcomings normally associated with suboptimization should be considerably less than if both subsystems are of equal stature. As a result of high recovery rates (estimated at 90 - 95%) being experienced on reparable items, the repaired item subsystem addressed by this paper would be proportionally larger (in terms of items controlled) than the purchased item subsystem. Under these conditions, the savings that would be realized by optimizing the total system might well be less than the costs involved in developing and operating under the more complex ordering rules required by total system optimizing techniques. Clearly, a trade-off does exist and a study to better establish this relationship would serve to channel future modeling efforts into the most promising direction.

4. CONCLUSION

This report has dealt with the study of an inventory system in which both procurement and repair decision rules were to be determined. The models presented represent a wide variety of assumptions about the manner in which the system is operated: continuous repair and batch repair, continuous review and periodic review, a single inventory of RFI items which are indistinguishable as to their source, and dual inventories of procured RFI items and repaired RFI items operating independently, etc. We have presented both viewpoints with regard to the objective function for such a system: minimize total variable costs after postulating a shortage cost, and minimize shortages subject to an overall budget constraint on variable costs. No single model or formulation is seen to be <u>the</u> best approach; all are approximate models whose assumptions are more or less valid depending upon the characteristics of the reparable item in question. The conclusions of each of the models speak for themselves.

Still some general observations, conclusions, and questions of applicability remain to be discussed.

The first observation is that the recovery rate is the governing parameter in a reparable system. The importance of considering procurement and repair as joint functions within a reparable <u>system</u> and the importance of optimal procurement decision rules vary inversely with the recovery rate. If the recovery rate is near unity, procurement and its impact on the budget become insignificant. Some further comments about the recovery rate are thus appropriate.

The recovery rate postulated in this report is the overall recovery rate; the historical percentage of the items which were demanded in RFI condition, used, and successfully repaired. In fact, there are two points within the system where an item may be scrapped. Items may be scrapped by the user and never returned to the O&R. This can occur if the user judges the item to be beyond repair or if the item is lost. Further, the O&R may scrap items returned by the user if a successful repair cannot be economically effected. Data on overall recovery rates were not obtained. However, the data of a not too recent, two-year study of O&R recovery rates were obtained [25] . A random sample of 120 items indicates an average O&R recovery rate of 93.3%. Some sample items and their O&R recovery rates are: gyro horizon indicators, 98%; accelerometers, 96%; transmitters, 92%; cameras, 87%; oil coolers, 85%; and manifolds, 82%. Overall recovery rates are, of course, somewhat lower; how much lower is not known. Thus, procurement may be quite important for some reparables and insignificant for others, depending upon the item recovery rate.

It also seems clear that the importance of procurement varies with the item's demand stability. Procurement is very important for a reparable item phasing into the system and is non-existent for a reparable phasing out of the system. However, procurement may become relatively more important even for a stable item with a high recovery rate if the general level of operations is increased dramatically (such as is now the case in Southeast Asia). A conclusion is that the importance of procurement in a reparable item inventory system varies with circumstances which are both item and time dependent.

Since, in any case, repair is relatively more important than procurement as a source of RFI items, improvements in the repair turn-around time will yield the greatest single benefit to the system. This conclusion is reinforced by the fact that the transit portions of the complete cycle of a reparable item seem to be minimal. Most reparables get premium transportation (air) from the user to the O&R and vice versa due to their high unit cost and generally essential nature. For a given protection level, reductions in the time an item spends in NRFI condition reduce the inventory required in the system. It is estimated that for every week of time in the repair cycle time, the Navy must own about $12 million of additional inventory of reparable items, not including engines, [26]. It is clear that the repair process is an area where further study will yield great benefits.

In this regard, it is noted that the repair of reparable items may be accomplished at any of three levels: at the user level, at an intermediate level, or at the major level (O&R). Only repair at the O&R level has been considered here. Lower level repair should also be studied as it influences the total reparable's inventory system.

Finally, we address the question of the applicability and usefulness of the models of this report. In general, the models are not immediately useful for inventory control within the Supply Corps. A number of factors either limit their application or require further study before application would be possible. All of the probabilistic models require iterative solutions for the decision variables. Due to computer limitations and the large number of items stocked in the real system, models requiring iterative solutions are not practicable. However, it is probable that suitable approximations could be developed to yield solutions directly.

A factor of the problem which requires further study is the distribution of the return of NRFI items to the O&R and its relationship (or lack of it) to the demand pattern. The probabilistic models of Section 3 have each made assumptions here, from complete independence to complete dependence. Much more should be understood about the carcass return process.

The greatest weakness of the models, with respect to current Supply Corps operations, however, is in the area of budget constraints. One of the models of Section 3 explicitly postulates a budget constraint and minimizes shortages subject to the budget constraint through a

Lagrangian function. The other models postulated a shortage cost and simply minimized costs. These approaches are equivalent if one is willing to parameterize the shortage cost and iterate until the shortage cost is found which will just satisfy the budget.

But, in all the models, the assumption was made that a higher authority had allocated a portion of some overall budget to the item in question. In other words, for a given item there was a single monitary constraint and the amounts spent for procurement, repair, writing orders, etc., would be obtained as by-products of the optimal solution. Such is not the case in current operations where there is a very complex funding structure. The purchase of a new reparable item may be charged to either the Navy Stock Fund or the Appropriation Purchases Account. The day-to-day operations of the Inventory Control Point are funded by an Operations and Maintenance Fund. It is this fund which determines the size of the clerical and supervisory staff at the ICP and thus constraints the number of procurement orders which can be executed in any fiscal period. Finally, the repair of a reparable item carcass is charged to the funds for operating the O&R. The O&R operating funds come from NAVAIR, NAVSHIPS, or other commands.

Additional funding problems could also be listed. In any case, it is clear that the complex funding structure and the number of items involved create problems which are not successfully handled in the models of this report. A point we wish to make, though, is that the optimal solutions "should" determine the budget allocations and not vice versa.

In conclusion, it is hoped that the research which this report describes does increase the Navy's understanding of reparable item inventory systems and, subject to the limitations noted above, is useful in the control of such systems.

APPENDIX A

The portion of the total demand being satisfied from the purchased-item inventory I_1 is $(1 - r)X$, where X is a normally distributed random variable with parameters μ_X and σ_X^2. As shown in reference [24], the probability density function for $(1 - r)X$ is given by

$$F_{(1-r)X}(y) = \frac{1}{1-r} f_X\left(\frac{y}{1-r}\right) \quad . \qquad (A-1)$$

From (A - 1), it follows that $(1 - r)X$ is a normally distributed random variable with parameters:

$$\text{mean} = \mu_X (1 - r) \quad ,$$

$$\text{variance} = \sigma_X^2 (1 - r)^2 \quad .$$

It was shown in subsection 2.2 that

$$I_1 = H_1 - \sum_{i=1}^{L+1} (1 - r) X_i \quad . \qquad (A-2)$$

Let

$$Z = \sum_{i=1}^{L+1} (1 - r) X_i \quad .$$

Due to the assumed independence of the X_i's, Z is also normally distributed with parameters:

$$\mu_Z = (L + 1) \mu_X (1 - r) \quad ,$$

$$\sigma_Z^2 = (L + 1) \sigma_X^2 (1 - r)^2 \quad . \qquad (A-3)$$

The distribution function $F(x)$ of the random variable I_1 is determined as follows:

$$
\begin{aligned}
F_{I_1}(x) &= P[I_1 \le x] \\
&= P[H_1 - Z \le x] \\
&= P[Z \ge H_1 - x] \\
&= 1 - F_Z(H_1 - x) \\
&= 1 - \int_{-\infty}^{H_1 - x} f_Z(z)\, dz \quad . \qquad \text{(A - 4)}
\end{aligned}
$$

In terms of a standard normal density function $\phi(y)$, equation (A - 4) can be written as

$$
F_{I_1}(x) = 1 - \int_{-\infty}^{\frac{H_1 - x - \mu_Z}{\sigma_Z}} \phi(y)\, dy \quad .
$$

Therefore,

$$
\begin{aligned}
f_{I_1}(x) &= \frac{1}{\sigma_Z} \phi\left(\frac{H_1 - x - \mu_Z}{\sigma_Z}\right) \\
&= \frac{1}{\sigma_Z} \phi\left(\frac{x - H_1 + \mu_Z}{\sigma_Z}\right) \\
&= \frac{1}{\sqrt{2\pi}\, \sigma_Z} e^{-\frac{1}{2\sigma_Z^2}[x - (H_1 - \mu_Z)]^2} \quad .
\end{aligned}
$$

It follows that I_1 is normally distributed with the following parameters (replacing μ_Z and σ_Z with their equivalent (A - 3) forms):

$$\mu_1 = H_1 - (L + 1)(1 - r)\mu_X \quad ,$$

$$\sigma_1^2 = (L + 1)(1 - r)^2 \sigma_X^2 \quad .$$

APPENDIX B

The covariance of I_1 and I_2, $\mathrm{Cov}(I_1, I_2)$, is derived as follows:

$$\mathrm{Cov}(I_1, I_2) = E[I_1 I_2] - E[I_1]\,E[I_2] \quad , \qquad \text{(B-1)}$$

where $E[X] \equiv$ expected value of X.

From equations (3) and (11), we can write

$$E[I_1 I_2] = E\Big[\Big(H_1 - \sum_{i=1}^{L+1} (1-r) X_i\Big) \cdot \Big(H_2 - \sum_{i=1}^{R+K^1+1} r X_i\Big)\Big]$$

$$= H_1 H_2 - H_2 (L+1)(1-r)\mu_X - H_1 (R + K^1 + 1) r \mu_X$$

$$+ (1-r) r \; E\Big[\sum_{i=1}^{L+1} X_i \sum_{i=1}^{R+K^1+1} X_i\Big] \quad .$$

Based on the condition that $L \geq R + K^1$,

$$E\Big[\sum_{i=1}^{L+1} X_i \sum_{i=1}^{R+K^1+1} X_i\Big] = E\Big[\sum_{i=1}^{R+K^1+1} X_i^2\Big] + E\Big[\sum_{i<j}^{R+K^1+1}\sum 2 X_i X_j\Big]$$

$$+ E\Big[\sum_{i=1}^{R+K^1+1} \sum_{j=R+K^1+2}^{L+1} X_i X_j\Big]$$

$$= (R + K^1 + 1)(\sigma_X^2 + \mu_X^2) + (R + K^1) \cdot (R + K^1 - 1)\mu_X^2 + (R + K^1 + 1)(L - R - K^1)\mu_X^2$$

$$= (R + K^1 + 1)(\sigma_X^2 + (L+1)\mu_X^2) \quad .$$

Therefore,

$$E[I_1 I_2] = H_1 H_2 - H_2 (L+1)(1-r)\mu_X - H_1 (R+K^1+1) r \mu_X$$
$$+ (1-r) r (R+K^1+1) (\sigma_X^2 + (L+1)\mu_X^2) . \qquad (B-2)$$

Now,

$$E[I_1] E[I_2] = \mu_1 \mu_2 = (H_1 - (L+1)(1-r)\mu_X)$$
$$\cdot (H_2 - (R+K^1+1) r \mu_X) . \qquad (B-3)$$

Putting the (B - 2) and (B - 3) forms of $E[I_1 I_2]$ and $E[I_1] E[I_2]$ back into equation (B - 1) and clearing terms gives

$$\mathrm{Cov}(I_1, I_2) = r(1-r)(R+K^1+1)\sigma_X^2 . \qquad (B-4)$$

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Reparable items, as opposed to consumable items, are usually rebuilt upon removal from service. A reparable item inventory system is composed of an inventory of ready-for-issue (RFI) items and an inventory of non-ready-for-issue (NRFI) items awaiting repair at the overhaul and repair facility. Since not all units issued in RFI condition will be recovered, procurement is necessary to supplement repair and maintain the population of units. This report describes such a system and presents a number of mathematical models, both deterministic and probabilistic, which prescribe the manner in which such a system should be operated.

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